Confidential

Department of Examinations - Sri Lanka

G.C.E. (A/L) Examination - 2017

10 - Combined Mathematics I

Marking Scheme

This has been prepared for the use of marking examiners. Changes would be made according to the views presented at the Chief/Assistant Examiners' meeting.

G.C.E. (A/L) Examination - 2017

10 - Combined Mathematics

Distribution of Marks

Paper I :

Part A :	10 x 25	= 250
Part B :	05 x 150	= 750

Total **= 1000/10**

Paper 1- Final Mark **= 100**

1. Using the **Principle of Mathematical Induction**, prove that $\sum_{r=1}^{n} r(3r+1) = n(n+1)^2$ for all $n \in \mathbb{Z}^+$.

For $n=1$, L.H.S. $= 1\cdot(3+1) = 4$ and R.H.S. $= 1\cdot(1+1)^2 = 4$. (5)

$\therefore$ The result is true for $n=1$.

Take any $p \in \mathbb{Z}^+$ and assume that the result is true for $n = p$.

i.e. $\sum_{r=1}^{p} r(3r+1) = p(p+1)^2$. ---------------- (1) (5)

Now $\sum_{r=1}^{p+1} r(3r+1) = \sum_{r=1}^{p} r(3r+1) + (p+1)(3p+4)$ (5)

$= p(p+1)^2 + (p+1)(3p+4)$

$= (p+1)(p^2 + p + 3p + 4)$

$= (p+1)(p+2)^2$. (5)

Hence if the result is true for $n = p$, then it is also true for $n = p+1$. We have already proved that the result is true for $n=1$.

Hence by the Principle of Mathematical Induction, the result is true for all $n \in \mathbb{Z}^+$.

(5)

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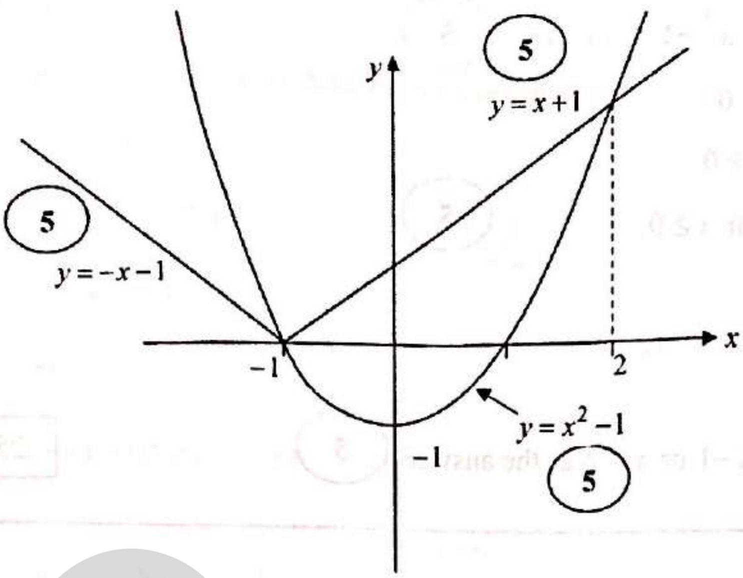
2. Find all real values of x satisfying the inequality $x^2 - 1 \geq |x+1|$.



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At the points of intersection, we must have $x \geq -1$ and $x^2 - 1 = x + 1$, and so $x = -1$ or $x = 2$.

The solutions are the values of x satisfying $x \leq -1$ or $x \geq 2$. (5) 25

Aliter 1

$$|x+1| = \begin{cases} x+1 & \text{if } x \geq -1 \\ -(x+1) & \text{if } x < -1 \end{cases}$$

Case (i) $x \geq -1$

In this case, $x^2 - 1 \geq |x+1| \Leftrightarrow x^2 - 1 \geq x + 1$ (5)

$\Leftrightarrow x^2 - x - 2 \geq 0$

$\Leftrightarrow (x+1)(x-2) \geq 0$

$\Leftrightarrow x \leq -1$ or $x \geq 2$. (5)

Since $x \geq -1$, the solutions are $x = -1$ or $x \geq 2$.

Case (ii) $x < -1$,

In this case, $x^2 - 1 \geq |x+1| \Leftrightarrow x^2 - 1 \geq -(x+1)$ (5)

$\Leftrightarrow x^2 + x \geq 0$

$\Leftrightarrow x(x+1) \geq 0$

$\Leftrightarrow x \leq -1$ or $x \geq 0$. (5)

Since $x < -1$, the solutions are $x < -1$.

From the two cases, we get $x \leq -1$ or $x \geq 2$ as the answer. (5)

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Aliter 2

Case (i) $x > -1$

$x^2 - 1 \geq |x+1| \Leftrightarrow x^2 - 1 \geq x + 1$ (5)

$\Leftrightarrow x \leq -1$ or $x \geq 2$. (5)

Since $x > -1$, the solutions are $x \geq 2$.

Case (ii) $x \leq -1$

$x^2 - 1 \geq |x+1| \Leftrightarrow x^2 - 1 \geq -(x+1)$ (5)

$\Leftrightarrow x \leq -1$ or $x \geq 0$. (5)

Since $x \leq -1$, the solutions are $x \leq -1$.

From the two cases, we get $x \leq -1$ or $x \geq 2$ as the answer. (5)

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Aliter 3

Case (i) $x^2 \geq 1$

In this case $x^2 - 1 \geq 0$, and so both sides are non-negative.

$\therefore\ x^2 - 1 \geq |x+1|$

$\Leftrightarrow\ (x^2-1)^2 \geq (x+1)^2$ (5)

$\Leftrightarrow\ (x+1)^2(x-1)^2 - (x+1)^2 \geq 0$

$\Leftrightarrow\ (x+1)^2[(x-1)^2 - 1] \geq 0$

$\Leftrightarrow\ (x+1)^2 x(x-2) \geq 0$ (5)

$\Leftrightarrow\ x = -1$ or $x \leq 0$ or $x \geq 2$ (5)

Since $x^2 \geq 1 \Leftrightarrow x \leq -1$ or $x \geq 1$, the solutions are $x \leq -1$ or $x \geq 2$. (5)

Case (ii) $x^2 < 1$

Since $x^2 - 1 < 0$, and hence there are no solution. From the two cases, we get $x \leq -1$ or $x \geq 2$ as the answer. (5)

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5. Let $0 < \alpha < \frac{\pi}{2}$. Show that $\lim_{x \to \alpha} \frac{x^3 - \alpha^3}{\tan x - \tan \alpha} = 3\alpha^2 \cos^2 \alpha$.

(5)

$$\lim_{x \to \alpha} \frac{x^3 - \alpha^3}{\tan x - \tan \alpha} = \lim_{x \to \alpha} \frac{(x-\alpha)(x^2 + \alpha x + \alpha^2)}{\frac{\sin x}{\cos x} - \frac{\sin \alpha}{\cos \alpha}}$$

$$= \lim_{x \to \alpha} \frac{(x-\alpha)\cos x \cos \alpha \cdot (x^2 + \alpha x + \alpha^2)}{\sin x \cos \alpha - \cos x \sin \alpha}$$ (5)

$$= \lim_{x \to \alpha} \frac{x-\alpha}{\sin(x-\alpha)} \cdot \cos x \cos \alpha \cdot (x^2 + \alpha x + \alpha^2)$$ (5)

(5)

$$= 1 \cdot \cos \alpha \cdot \cos \alpha \cdot (3\alpha^2)$$

$$= 3\alpha^2 \cos^2 \alpha.$$ (5)

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Aliter 1

$$\lim_{x \to \alpha} \frac{x^3 - \alpha^3}{\tan x - \tan \alpha} = \lim_{x \to \alpha} \frac{(x-\alpha)(x^2 + \alpha x + \alpha^2)}{\tan(x-\alpha)(1 + \tan x \tan \alpha)}$$ (5) $$\left(\because \tan(x-\alpha) = \frac{\tan x - \tan \alpha}{1 + \tan x \tan \alpha} \right)$$

$$= \lim_{x \to \alpha} \frac{x-\alpha}{\tan(x-\alpha)} \cdot \frac{x^2 + \alpha x + \alpha^2}{(1 + \tan x \tan \alpha)}$$ (5)

$$= \lim_{x \to \alpha} \frac{x-\alpha}{\sin(x-\alpha)} \cdot \frac{\cos(x-\alpha) \cdot (x^2 + \alpha x + \alpha^2)}{(1 + \tan x \tan \alpha)}$$

(5)

$$= 1 \cdot \frac{1 \cdot 3\alpha^2}{1 + \tan^2 \alpha}$$ (5)

$$= \frac{3\alpha^2}{\sec^2 \alpha} = 3\alpha^2 \cos^2 \alpha.$$ (5)

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Aliter 2

$$\lim_{x\to\alpha}\frac{x^3-\alpha^3}{\tan x-\tan\alpha}=\lim_{x\to\alpha}\frac{x^3-\alpha^3}{x-\alpha}\cdot\frac{x-\alpha}{\frac{\sin x}{\cos x}-\frac{\sin\alpha}{\cos\alpha}}$$ (5)

$$=\lim_{x\to\alpha}\frac{x^3-\alpha^3}{x-\alpha}\cdot\frac{x-\alpha}{\frac{\sin x\cos\alpha-\cos x\sin\alpha}{\cos x\cos\alpha}}$$ (5)

$$=\lim_{x\to\alpha}\frac{x^3-\alpha^3}{x-\alpha}\cdot\frac{(x-\alpha)}{\sin(x-\alpha)}\cdot\cos x\cos\alpha$$ (5)

$$=3\alpha^2\cdot 1\cdot\cos^2\alpha$$

(5)

$$=3\alpha^2\cos^2\alpha.$$ (5)

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6. Let $0<a<b$. Show that $\frac{d}{dx}\sin^{-1}\left(\sqrt{\frac{b-a}{b}}\cos x\right)=-\frac{\sqrt{b-a}\sin x}{\sqrt{a\cos^2 x+b\sin^2 x}}$.

Hence, find $\int\frac{\sin x}{\sqrt{a\cos^2 x+b\sin^2 x}}dx$.

$$\frac{d}{dx}\sin^{-1}\left(\sqrt{\frac{b-a}{b}}\cos x\right)=\frac{1}{\sqrt{1-\frac{(b-a)}{b}\cos^2 x}}\times\sqrt{\frac{b-a}{b}}\times(-\sin x)$$ (5) + (5)

$$=-\frac{\sin x}{\sqrt{b-b\cos^2 x+a\cos^2 x}}\times\sqrt{b-a}$$

$$=-\frac{\sqrt{b-a}\sin x}{\sqrt{a\cos^2 x+b\sin^2 x}}$$ (5)

$$\therefore\int-\frac{\sqrt{b-a}\sin x}{\sqrt{a\cos^2 x+b\sin^2 x}}dx=\sin^{-1}\left(\sqrt{\frac{b-a}{b}}\cos x\right)+\text{constant}$$ (5)

$$\int \frac{\sin x}{\sqrt{a\cos^2 x + b\sin^2 x}}\,dx = -\frac{1}{\sqrt{b-a}}\sin^{-1}\left(\sqrt{\frac{b-a}{b}}\cos x\right) + C,$$ where C is an arbitrary constant. (5)

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Aliter

Let $y = \sin^{-1}\left(\sqrt{\frac{b-a}{b}}\cos x\right)$.

Then $\sin y = \sqrt{\frac{b-a}{b}}\cos x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

$$\cos y \frac{dy}{dx} = \sqrt{\frac{b-a}{b}}(-\sin x) \text{----------------- (1)}$$ (5)

$$\cos y = \sqrt{1-\sin^2 y} \quad \left(\because -\frac{\pi}{2} \le y \le \frac{\pi}{2}\right)$$

$$= \sqrt{1-\frac{b-a}{b}\cos^2 x}$$

$$= \sqrt{\frac{b(1-\cos^2 x)+a\cos^2 x}{b}}$$

$$= \frac{\sqrt{a\cos^2 x + b\sin^2 x}}{\sqrt{b}}$$ (5)

$$\therefore (1) \Rightarrow \frac{dy}{dx} = -\frac{\sqrt{b-a}\sin x}{\sqrt{a\cos^2 x + b\sin^2 x}}.$$ (5)

Integration as before. (10)

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7. A curve C is given parametrically by $x=3\cos\theta-\cos^3\theta$, $y=3\sin\theta-\sin^3\theta$ for $0<\theta<\frac{\pi}{2}$.
Show that $\frac{dy}{dx}=-\cot^3\theta$.
Find the coordinates of the point P on the curve C at which the gradient of the tangent line is -1.

$x=3\cos\theta-\cos^3\theta \qquad y=3\sin\theta-\sin^3\theta$

$\frac{dx}{d\theta}=-3\sin\theta+3\cos^2\theta\sin\theta$; (5) $\quad \frac{dy}{d\theta}=3\cos\theta-3\sin^2\theta\cos\theta$ (5)

$$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{3\cos\theta(1-\sin^2\theta)}{-3\sin\theta(1-\cos^2\theta)}=-\frac{\cos^3\theta}{\sin^3\theta}=-\cot^3\theta.$$ (5)

$$\frac{dy}{dx}=-1\Leftrightarrow\cot\theta=1\Leftrightarrow\theta=\frac{\pi}{4}$$ (5)

$$P\equiv\left(\frac{3}{\sqrt{2}}-\frac{1}{2\sqrt{2}},\frac{3}{\sqrt{2}}-\frac{1}{2\sqrt{2}}\right)\equiv\left(\frac{5}{2\sqrt{2}},\frac{5}{2\sqrt{2}}\right).$$ (5)

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8. Let l_1 and l_2 be the straight lines given by $3x-4y=2$ and $4x-3y=1$ respectively.
(i) Write down the equations of the bisectors of the angles between l_1 and l_2.
(ii) Find the equation of the bisector of the acute angle between l_1 and l_2.

Bisectors are given by

$$\frac{3x-4y-2}{5}=\pm\frac{4x-3y-1}{5}$$ (5)

$x+y+1=0$ or $7x-7y-3=0$ (5)

Let α be the acute angle between l_1 and $x+y+1=0$

$$\tan\alpha=\left|\frac{\frac{3}{4}+1}{1-\frac{3}{4}}\right|$$ (5)

$=7>1$. (5)

$\therefore 7x-7y-3=0$ is the bisector of the acute angle between l_1 and l_2. (5)

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9. Let S be the circle given by $x^2 + y^2 - 4 = 0$ and let l be the straight line given by $y = x + 1$. Find the equation of the circle which passes through the points of intersection of S and l, and also intersects the circle S orthogonally.

The required equation has the form $(x^2+y^2-4)+\lambda(y-x-1)=0$, where $\lambda \in \mathbb{R}$.

i.e. $x^2+y^2-\lambda x+\lambda y-\lambda-4=0$. (10)

If this is orthogonal to S, with $g=0; f=0; c=-4; g'=-\frac{\lambda}{2}; f'=\frac{\lambda}{2}; c'=-\lambda-4$,

we must have $2gg'+2ff'=c+c'$. (5)

i.e. $0=-\lambda-8$

$\therefore \lambda=-8$. (5)

$\therefore$ The answer is $x^2+y^2+8x-8y+4=0$. (5)

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10. Show that $\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^2 = 1+\sin\theta$ for $-\pi<\theta\le\pi$. Hence, show that $\cos\frac{\pi}{12}+\sin\frac{\pi}{12}=\sqrt{\frac{3}{2}}$ and also find the value of $\cos\frac{\pi}{12}-\sin\frac{\pi}{12}$. Deduce that $\sin\frac{\pi}{12}=\frac{\sqrt{3}-1}{2\sqrt{2}}$.

$$\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2=\sin^2\frac{\theta}{2}+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}+\cos^2\frac{\theta}{2}$$ (5)

$$=1+\sin\theta \quad \left(\because \sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}=1 \text{ and } 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=\sin\theta.\right)$$

Let $\theta=\frac{\pi}{6}$: (5)

Then $\left(\cos\frac{\pi}{12}+\sin\frac{\pi}{12}\right)^2=1+\frac{1}{2}$.

$\therefore \sin\frac{\pi}{12}+\cos\frac{\pi}{12}=\sqrt{\frac{3}{2}}$ ------------- (1) (5) $\left(\because \sin\frac{\pi}{12}+\cos\frac{\pi}{12}>0\right)$

Let $\theta = \frac{-\pi}{6}$:

Then $\left(\cos\frac{\pi}{12} - \sin\frac{\pi}{12}\right)^2 = \frac{1}{2}$.

$\therefore \cos\frac{\pi}{12} - \sin\frac{\pi}{12} = \frac{1}{\sqrt{2}}$ ------------- (2) ($\because \sin\frac{\pi}{12} < \cos\frac{\pi}{12}$) (5)

$(1) - (2) \Rightarrow \sin\frac{\pi}{12} := \frac{\sqrt{3}-1}{2\sqrt{2}}$. (5)

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Part B

11.(a) Let $f(x) = 3x^2 + 2ax + b$, where $a, b \in \mathbb{R}$.

It is given that the equation $f(x) = 0$ has two real distinct roots. Show that $a^2 > 3b$.

Let α and β be the roots of $f(x)=0$. Write down $\alpha+\beta$ in terms of a and $\alpha\beta$ in terms of b.

Show that $|\alpha - \beta| = \frac{2}{3}\sqrt{a^2 - 3b}$.

Show further that the quadratic equation with $|\alpha + \beta|$ and $|\alpha - \beta|$ as its roots is given by $9x^2 - 6\left(|a| + \sqrt{a^2 - 3b}\right)x + 4\sqrt{a^4 - 3a^2b} = 0$.

(b) Let $g(x) = x^3 + px^2 + qx + 1$, where $p, q \in \mathbb{R}$. When $g(x)$ is divided by $(x-1)(x+2)$, the remainder is $3x+2$. Show that the remainder when $g(x)$ is divided by $(x-1)$ is 5, and that the remainder when $g(x)$ is divided by $(x+2)$ is -4.

Find the values of p and q, and show that $(x+1)$ is a factor of $g(x)$.

(5)

(a) The discriminant $\Delta = (2a)^2 - 4(3)(b)$

$= 4(a^2 - 3b)$. (5)

Since $f(x) = 0$ has two real distant roots, we must have $\Delta > 0$. (5)

$\therefore \; a^2 > 3b$. (5) **20**

$\alpha + \beta = -\frac{2a}{3}$ and $\alpha\beta = \frac{b}{3}$.

(5) (5) **10**

$$(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$$ (10)

$$=\frac{4a^2}{9}-\frac{4b}{3}$$ (5)

$$=\frac{4}{9}(a^2-3b).$$ (5)

$$\therefore\ |\alpha-\beta|=\frac{2}{3}\sqrt{a^2-3b}.$$ (5)

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Let $\alpha'=|\alpha+\beta|$ and $\beta'=|\alpha-\beta|$.

Then $\alpha'=\frac{2}{3}|a|$ and $\beta'=\frac{2}{3}\sqrt{a^2-3b}$.

(5)

The required equation is $(x-\alpha')(x-\beta')=0$. (5)

i.e. $x^2-(\alpha'+\beta')x+\alpha'\beta'=0$. (5)

$$\Rightarrow x^2-\left(\frac{2}{3}|a|+\frac{2}{3}\sqrt{a^2-3b}\right)x+\frac{4}{9}|a|\sqrt{a^2-3b}=0.$$

(5) (5)

$$\Rightarrow 9x^2-6\left(|a|+\sqrt{a^2-3b}\right)x+4\sqrt{a^4-3a^2b}=0.$$ (5)

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(b) Since the remainder when $g(x)$ divided by $(x-1)(x+2)$ is $3x+2$, we have

$g(x)=h(x)(x-1)(x+2)+3x+2$, ------------ (1) (10)

where $h(x)$ is a polynomial of degree 1.

By the Remainder Theorem, the remainder when $g(x)$ is divided by $(x-1)$ is $g(1)$.

(5)

$(1) \Rightarrow g(1) = 5$. (5)

Hence, the remainder when $g(x)$ divided by $(x-1)$ is 5.

Again, by the Remainder Theorem, the remainder when $g(x)$ is divided by $(x+2)$ is $g(-2)$. (5)

$(1) \Rightarrow g(-2) = -4$. (5)

Hence, the remainder when $g(x)$ divided by $(x+2)$ is -4.

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$g(1) = 5 \Rightarrow 1 + p + q + 1 = 5$ (5)

$p + q = 3$

$g(-2) = -4 \Rightarrow -8 + 4p - 2q + 1 = -4$ (5)

$4p - 2q = 3$

$p = \frac{3}{2}$ and $q = \frac{3}{2}$.

(5) (5)

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(5) (5)

Now $g(-1) = -1 + p - q + 1 = 0. (\because p = q)$

Thus, by the Factor Theorem, $(x+1)$ is a factor of $g(x)$. (5)

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12. (*a*) Write down the binomial expansion of $(5+2x)^{14}$ in ascending powers of x.

Let T_r be the term containing x^r in the above expansion for $r = 0,1,2,\ldots,14$.

Show that $\frac{T_{r+1}}{T_r} = \frac{2(14-r)}{5(r+1)}x$ for $x \neq 0$.

Hence, find the value of r which gives the largest term of the above expansion, when $x = \frac{4}{3}$.

(*b*) Let $c \geq 0$. Show that $\frac{2}{(r+c)(r+c+2)} = \frac{1}{(r+c)} - \frac{1}{(r+c+2)}$ for $r \in \mathbb{Z}^+$.

Hence, show that $\sum_{r=1}^{n} \frac{2}{(r+c)(r+c+2)} = \frac{(3+2c)}{(1+c)(2+c)} - \frac{1}{(n+c+1)} - \frac{1}{(n+c+2)}$ for $n \in \mathbb{Z}^+$.

Deduce that the infinite series $\sum_{r=1}^{\infty} \frac{2}{(r+c)(r+c+2)}$ converges and find its sum.

By using this sum with suitable values for c, show that $\sum_{r=1}^{\infty} \frac{1}{r(r+2)} = \frac{1}{3} + \sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)}$.

(a) $(5+2x)^{14} = \sum_{r=0}^{14} {}^{14}C_r 5^{14-r}(2x)^r$ (10)

$= \sum_{r=0}^{14} {}^{14}C_r 5^{14-r} \cdot 2^r \cdot x^r$, where ${}^{14}C_r = \frac{14!}{r!(14-r)!}$ for $r = 0, 1, \ldots, 14$.

(5)

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Let $T_r = {}^{14}C_r 5^{14-r} \cdot 2^r \cdot x^r$ for $r = 0, 1, \ldots, 14$.

Then $\frac{T_{r+1}}{T_r} = \frac{14!\, 5^{13-r}\, 2^{r+1}}{(r+1)!(13-r)!} x^{r+1} \Big/ \frac{14!\, 5^{14-r}\, 2^r}{r!(14-r)!} x^r$ (5)

(10)

$= \frac{2(14-r)}{5(r+1)} x$ (5)

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Thus. $x=\frac{4}{3}\Rightarrow\frac{T_{r+1}}{T_r}=\frac{2}{5}\frac{(14-r)}{(r+1)}\cdot\frac{4}{3}$ (5)

Hence, $\frac{T_{r+1}}{T_r}\underset{<}{\geq}1$ according as $\frac{8}{15}\frac{(14-r)}{(r+1)}\underset{<}{\geq}1$.

(5) (5)

i.e. according as $112-8r\underset{<}{\geq}15r+15$.

i.e. according as $r\underset{>}{\leq}\frac{97}{23}=4\frac{5}{23}$.

(5)

$T_0<T_1<T_2<T_3<T_4<T_5>T_6\cdots>T_{14}$ (10)

$\therefore$ The required value is $r=5$. (5)

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(b) $\frac{1}{r+c}-\frac{1}{r+c+2}=\frac{(r+c+2)-(r+c)}{(r+c)(r+c+2)}$ (5)

$=\frac{2}{(r+c)(r+c+2)}$. (5)

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Let $u_r=\frac{2}{(r+c)(r+c+2)}$ for $r\in\mathbb{Z}^+$

Then

$r=1;\quad u_1=\frac{1}{1+c}-\frac{1}{3+c}$

(5)

$r=2;\quad u_2=\frac{1}{2+c}-\frac{1}{4+c}$

$r=3;\quad u_3=\frac{1}{3+c}-\frac{1}{5+c}$ (5)

$\vdots$

$r = n-2;\ u_{n-2} = \frac{1}{n-2+c} - \frac{1}{n+c}$ (5)

$r = n-1;\ u_{n-1} = \frac{1}{n-1+c} - \frac{1}{n+c+1}$ (5)

$r = n;\ u_n = \frac{1}{n+c} - \frac{1}{n+c+2}$

$$\sum_{r=1}^{n} u_r = \frac{1}{1+c} + \frac{1}{2+c} - \frac{1}{n+c+1} - \frac{1}{n+c+2}$$ (10)

$$= \frac{3+2c}{(1+c)(2+c)} - \frac{1}{n+c+1} - \frac{1}{n+c+2}$$ (5)

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The limit of the R.H.S. as $n \to \infty$ is $\frac{3+2c}{(1+c)(2+c)}$. (10)

$\therefore \sum_{r=1}^{\infty} u_r$ convergent and the sum is $\frac{3+2c}{(1+c)(2+c)}$.

(5) (5)

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Put $c=0: \sum_{r=1}^{\infty} \frac{1}{r(r+2)} = \frac{3}{4}$. ----------------(1) (5)

Put $c=1: \sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)} = \frac{5}{12}$. (5)

$\Rightarrow \frac{1}{3} + \sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)} = \frac{1}{3} + \frac{5}{12} = \frac{3}{4}$ ----------------(2)

Now, (1) and (2) $\Rightarrow \sum_{r=1}^{\infty} \frac{1}{r(r+2)} = \frac{1}{3} + \sum_{r=1}^{\infty} \frac{1}{(r+1)(r+3)}$. (5)

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13.(a) Let $\mathbf{A} = \begin{pmatrix} 2 & a & 3 \\ -1 & b & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 & a \\ 1 & b & 0 \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix}$, where $a, b \in \mathbb{R}$.

It is given that $\mathbf{AB}^T = \mathbf{P}$, where $\mathbf{B}^T$ denotes the transpose of the matrix $\mathbf{B}$. Show that $a = 1$ and $b = -1$, and with these values for a and b, find $\mathbf{B}^T\mathbf{A}$.

Write down $\mathbf{P}^{-1}$, and using it, find the matrix $\mathbf{Q}$ such that $\mathbf{PQ} = \mathbf{P}^2 + 2\mathbf{I}$, where $\mathbf{I}$ is the identity matrix of order 2.

(b) Sketch in an Argand diagram, the locus C of the points representing complex numbers z satisfying $|z| = 1$.

Let $z_0 = a(\cos\theta + i\sin\theta)$, where $a > 0$ and $0 < \theta < \frac{\pi}{2}$. Find the modulus in terms of a and the principal argument, in terms of θ, of each of the complex numbers $\frac{1}{z_0}$ and z_0^2.

Let P, Q, R and S be the points in the above Argand diagram representing the complex numbers z_0, $\frac{1}{z_0}$, $z_0 + \frac{1}{z_0}$ and z_0^2, respectively.

Show that when the point P lies on C above,

(i) the points Q and S also lie on C, and

(ii) the point R lies on the real axis between 0 and 2.

(a) $$AB^T = \begin{pmatrix} 2 & a & 3 \\ -1 & b & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & b \\ a & 0 \end{pmatrix} \quad \text{(5)}$$

$$= \begin{pmatrix} 2-a+3a & 2+ab \\ -1-b+2a & -1+b^2 \end{pmatrix} \quad \text{(10)}$$

$$AB^T = P \Leftrightarrow \begin{pmatrix} 2-a+3a & 2+ab \\ -1-b+2a & -1+b^2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix} \quad \text{(5)}$$

$$\Leftrightarrow 2+2a=4, \quad 2+ab=1, \quad -1+2a-b=2, \quad -1+b^2=0. \quad \text{(10)}$$

$$\Leftrightarrow a=1, \quad b=-1. \quad \text{(5)}$$

Now, $B^T A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & 3 \\ -1 & -1 & 2 \end{pmatrix}$ (5)

$= \begin{pmatrix} 1 & 0 & 5 \\ -1 & 0 & -5 \\ 2 & 1 & 3 \end{pmatrix}$. (5)

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$P^{-1} = -\frac{1}{2}\begin{pmatrix} 0 & -1 \\ -2 & 4 \end{pmatrix}$. (10)

Also, $PQ = P^2 + 2I \Leftrightarrow P^{-1}(PQ) = P^{-1}(P^2 + 2I)$ (5)

$\Leftrightarrow Q = P^{-1}P^2 + P^{-1}(2I)$ (5)

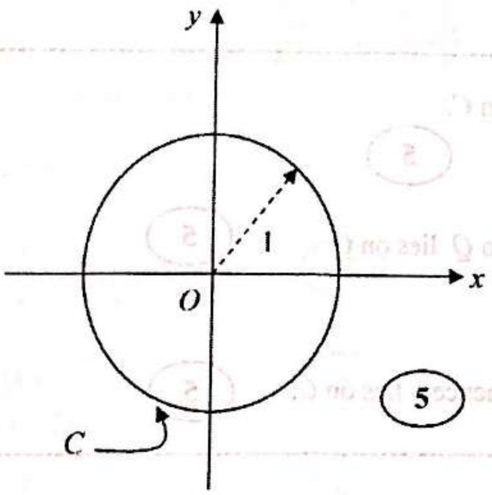
$\Leftrightarrow Q = P + 2P^{-1}$ (5)

$\Leftrightarrow Q = \begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 2 & -4 \end{pmatrix}$ (5)

$\therefore Q = \begin{pmatrix} 4 & 2 \\ 4 & -4 \end{pmatrix}$. (5)

35

(b)



(5)

5

First, $\frac{1}{z_0} = \frac{1}{a(\cos\theta + i\sin\theta)} \cdot \frac{(\cos\theta - i\sin\theta)}{(\cos\theta - i\sin\theta)}$ (5)

$= \frac{(\cos\theta - i\sin\theta)}{a(\cos^2\theta + \sin^2\theta)}$

$= \frac{1}{a}(\cos(-\theta) + i\sin(-\theta))$ (5)

Hence, $\left|\frac{1}{z_0}\right| = \frac{1}{a}$, and $\text{Arg}\left(\frac{1}{z_0}\right) = -\theta$.

(5) (5)

Next, $z_0{}^2 = a^2(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$

$= a^2\{(\cos^2\theta - \sin^2\theta) + 2i\cos\theta\sin\theta\}$ (5)

$= a^2(\cos 2\theta + i\sin 2\theta)$ (5)

Hence, $|z_0{}^2| = a^2$, and $\text{Arg}(z_0{}^2) = 2\theta$. (5)

(5) **40**

Suppose that P lies on C.

Then $a = 1$. (5)

$\therefore \left|\frac{1}{z_0}\right| = 1$ and so Q lies on C (5)

Also, $|z_0^2| = 1$ and hence S lies on C. (5)

15

$z_0 + \frac{1}{z_0} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$

$= 2\cos\theta$. (5)

Note that $0<\theta<\dfrac{\pi}{2}\Rightarrow 0<2\cos\theta<2$.

$\therefore$ The number represented by $z_0+\dfrac{1}{z_0}$ is real and lies between 0 and 2 on the real axis. (5)

10

14.(*a*) Let $f(x)=\dfrac{x^2}{(x-1)(x-2)}$ for $x\neq 1,2$.

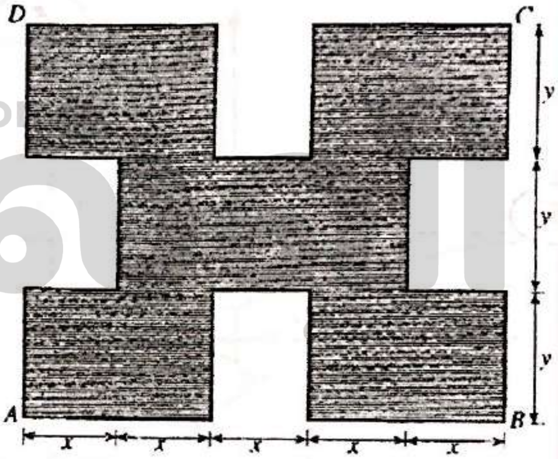
Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x)=\dfrac{x(4-3x)}{(x-1)^2(x-2)^2}$ for $x\neq 1,2$.

Sketch the graph of $y=f(x)$ indicating the asymptotes and the turning points.

Using the graph, solve the inequality $\dfrac{x^2}{(x-1)(x-2)}\leq 0$.

(*b*) The shaded region shown in the adjoining figure is of area 385 m². This region is obtained by removing four identical rectangles each of length y metres and width x metres from a rectangle $ABCD$ of length $5x$ metres and width $3y$ metres. Show that $y=\dfrac{35}{x}$ and that the perimeter P of the shaded region, measured in metres, is given by $P=14x+\dfrac{350}{x}$ for $x>0$.

Find the value of x such that P is minimum.



(a) $f(x)=\dfrac{x^2}{(x-1)(x-2)}$ for $x\neq 1,2$.

Then $f'(x)=\dfrac{(x-1)(x-2)2x-x^2(2x-3)}{(x-1)^2(x-2)^2}$ (10)

$=\dfrac{-6x^2+4x+3x^2}{(x-1)^2(x-2)^2}$ (5)

$=\dfrac{x(4-3x)}{(x-1)^2(x-2)^2}$ for $x\neq 1,2$. (5)

20

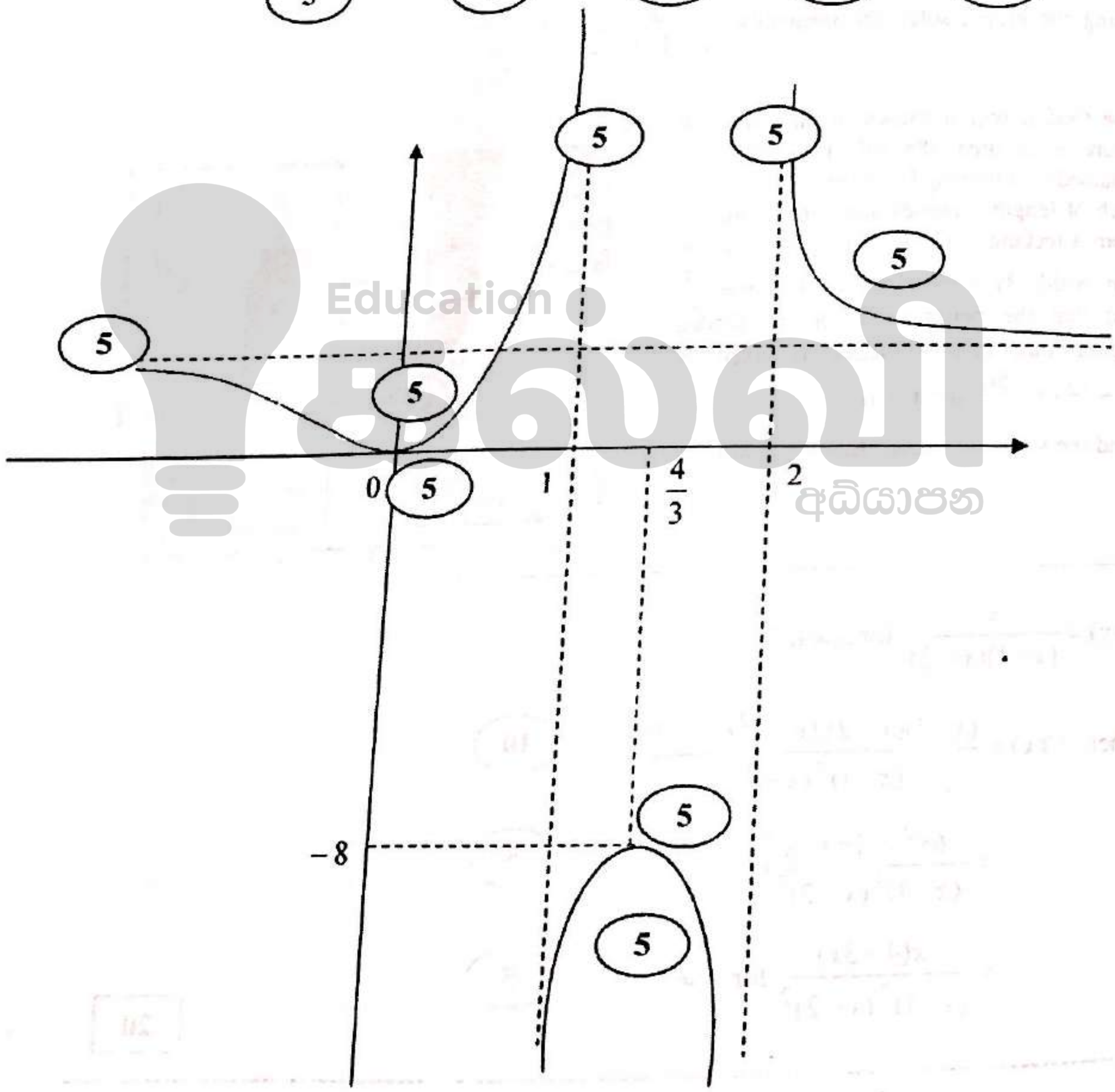
Horizontal Asymptote: $\lim_{x\to\pm\infty} f(x)=1$. Hence, it is $y=1$.

Note that $\lim_{x\to 1-} f(x)=\infty$ and $\lim_{x\to 1+} f(x)=-\infty$ $\lim_{x\to 2-} f(x)=-\infty$ and $\lim_{x\to 2+} f(x)=\infty$.

Vertical Asymptotes: $x=1,2$

$f'(x)=0 \Leftrightarrow x=0$ or $x=\frac{4}{3}$. (5)

	$-\infty<x<0$	$0<x<1$	$1<x<\frac{4}{3}$	$\frac{4}{3}<x<2$	$2<x<\infty$
Sign of $f'(x)$	(-)	(+)	(+)	(-)	(-)
	(5)	(5)	(5)	(5)	(5)



There are two turning points: $(0,0)$ - local minimum and $\left(\frac{4}{3}, -8\right)$ is a local maximum

(5) $x = 0$ or $2 < x < 2$ (5) **70**

(b) Area : $(5x)(3y) - 4xy = 385$ (5)

$11xy = 385$

$xy = 35$

$y = \frac{35}{x}$. (5)

Perimeter: $P = 2(5x + 3y) + 4x + 4y$ (5)

$= 14x + 10y$

$= 14x + \frac{350}{x}; \quad x > 0.$ (5)

$\frac{dP}{dx} = 14 - \frac{350}{x^2}$ (5)

$\frac{dP}{dx} = 0 \Leftrightarrow x^2 = \frac{350}{14} = 25$

(5)

$\therefore x = 5$ (5)

$\frac{dP}{dx} < 0$ for $0 < x < 5$ and $\frac{dP}{dx} > 0$ for $5 < x$

(5) (5)

$\therefore P$ is minimum when $x = 5$. (5) **50**

15.(a) (i) Express $\frac{1}{x(x+1)^2}$ in partial fractions and hence, find $\int \frac{1}{x(x+1)^2} dx$.

(ii) Using integration by parts, find $\int xe^{-x} dx$ and hence, find the area of the region enclosed by the curve $y = xe^{-x}$ and the straight lines $x = 1$, $x = 2$ and $y = 0$.

(b) Let $c > 0$ and $I = \int_0^c \frac{\ln(c+x)}{c^2+x^2} dx$. Using the substitution $x = c \tan\theta$, show that $I = \frac{\pi}{4c}\ln c + \frac{1}{c}J$, where $J = \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\, d\theta$.

Using the formula $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, where a is a constant, show that $J = \frac{\pi}{8}\ln 2$.

Deduce that $I = \frac{\pi}{8c}\ln(2c^2)$.

(i) $\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$ (10)

$1 = A(x+1)^2 + Bx(x+1) + Cx$

$1 = (A+B)x^2 + (2A+B+C)x + A$

By comparing coefficients,

$x^0 : 1 = A$

$x^1 : 0 = 2A + B + C$ (10)

$x^2 : 0 = A + B$

$\therefore$ $A = 1, B = -1$ and $C = -1$. (10)

$\int \frac{1}{x(x+1)^2} dx = \int \frac{1}{x} dx - \int \frac{1}{x+1} dx - \int \frac{1}{(x+1)^2} dx$ (5)

(15) $= \ln|x| - \ln|x+1| + \frac{1}{x+1} + C'$, where C' is an arbitrary constant.

50

(ii) $\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx$ (10)

$= -xe^{-x} - e^{-x} + C''$, where C'' is an arbitrary constant. (5)

(5)

Required area $= \int_1^2 xe^{-x}dx$ (5)

$= -(x+1)e^{-x}\Big|_1^2$ (5)

$= 2e^{-1} - 3e^{-2}.$ (5)

35

(b) Let $x = c\tan\theta$.

Then $dx = c\sec^2\theta\, d\theta$.

When $x = 0$, $\theta = 0$ and when $x = c, \theta = \frac{\pi}{4}$.

(5)

Thus, $I = \int_0^{\frac{\pi}{4}} \frac{\ln c(1+\tan\theta)}{c^2 + c^2\tan^2\theta} \cdot c\sec^2\theta\, d\theta$ (5) (5)

$= \int_0^{\frac{\pi}{4}} \frac{\ln c(1+\tan\theta)}{c^2\sec^2\theta} \cdot c\sec^2\theta\, d\theta$ (5)

$= \frac{1}{c}\int_0^{\frac{\pi}{4}} \{\ln c + \ln(1+\tan\theta)\} d\theta$ (5)

$= \frac{1}{c}\ln c \int_0^{\frac{\pi}{4}} d\theta + \frac{1}{c}\int_0^{\frac{\pi}{4}} \ln\{1+\tan\theta\} d\theta$

$$= \frac{1}{c}\ln c\cdot\theta\Big|_0^{\pi/4} + \frac{1}{c}J$$ (5)

$$= \frac{\pi}{4c}\ln c + \frac{1}{c}J.$$ (5)

35

$$J = \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta$$ (5)

$$= \int_0^{\frac{\pi}{4}} \ln\left\{1+\frac{1-\tan\theta}{1+\tan\theta}\right\}d\theta$$ (5)

$$= \int_0^{\frac{\pi}{4}} \ln\frac{2}{(1+\tan\theta)}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \{\ln 2 - \ln(1+\tan\theta)\}d\theta$$ (5)

$$= \ln 2\cdot\frac{\pi}{4} - J$$

$$\therefore J = \frac{\pi}{8}\ln 2.$$ (5)

$$\therefore I = \frac{\pi}{4c}\ln c + \frac{1}{c}\frac{\pi}{8}\ln 2$$ (5)

$$= \frac{\pi}{8c}\{2\ln c + \ln 2\}$$

$$= \frac{\pi}{8c}\ln(2c^2).$$ (5)

30

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16. Let $m \in \mathbb{R}$. Show that the point $P \equiv (0,1)$ does not lie on the straight line l given by $y = mx$.

Show that the coordinates of any point on the straight line through P perpendicular to l can be written in the form $(-mt,\ t+1)$, where t is a parameter.

Hence, show that the coordinates of the point Q, the foot of the perpendicular drawn from P to l, are given by $\left(\dfrac{m}{1+m^2}, \dfrac{m^2}{1+m^2}\right)$.

Show that, as m varies, the point Q lies on the circle S given by $x^2 + y^2 - y = 0$, and sketch the locus of Q in the xy-plane.

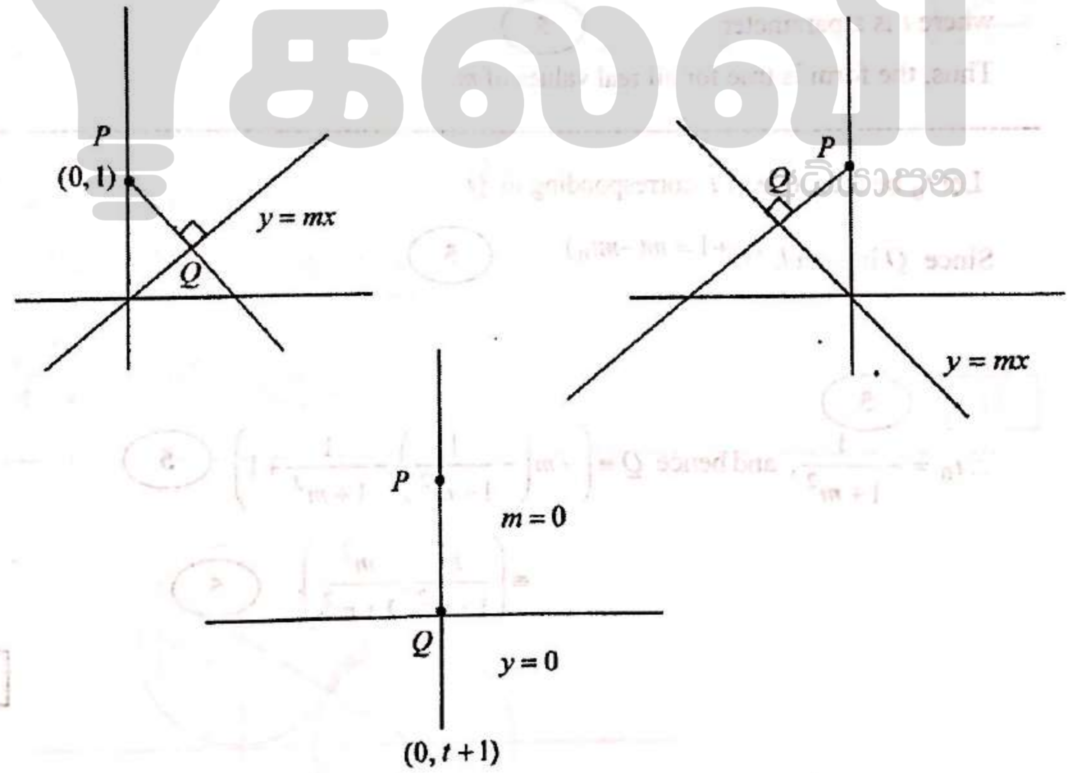
Also, show that the point $R \equiv \left(\dfrac{\sqrt{3}}{4}, \dfrac{1}{4}\right)$ lies on S.

Find the equation of the circle S' whose centre lies on the x-axis, and touches S externally at the point R.

Write down the equation of the circle having the same centre as that of S' and touching S internally.

If the point $(0,1)$ lies on l, then we must have $1 = m \times 0$. i.e. $1 = 0$, a contradiction. (5)

$\therefore (0,1)$ does not lie on l. (5) **10**



Case (i) : $m \neq 0$

In this case, the equation of the line through P perpendicular to l is given by

$y-1=-\frac{1}{m}(x-0)$. (10)

Let us introduce t into this equation by $y-1=-\frac{1}{m}(x-0)=t$ (say). (5)

Then $y=t+1$ and $x=-mt$, where t is a patameter.
(5) (5)

Hence, the coordinates of any point on the line through P perpendicular to l can be written in the form $(-mt, t+1)$, where t is a parameter.

Case (ii) : $m=0$

In this case, the equation of the line through P perpendicular to l is the y-axis and hence, the coordinates of any point on it can be written in the form $(0, t+1)$, where t is a parameter. (5)

Thus, the form is true for all real values of m.

30

Let t_0 be the value of t corresponding to Q.

Since Q lies on l, $t_0+1=m(-mt_0)$. (5)

(5)

$\therefore t_0=-\frac{1}{1+m^2}$, and hence $Q \equiv \left(-m\left(-\frac{1}{1+m^2}\right), -\frac{1}{1+m^2}+1\right)$ (5)

$\equiv \left(\frac{m}{1+m^2}, \frac{m^2}{1+m^2}\right)$. (5)

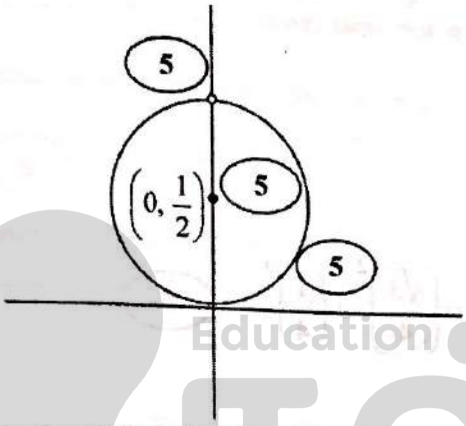
20

Put $x=\frac{m}{1+m^2}$ and $y=\frac{m^2}{1+m^2}$ in x^2+y^2-y: (5)

$$x^2+y^2-y=\frac{m^2}{(1+m^2)^2}+\frac{m^4}{(1+m^2)^2}-\frac{m^2}{1+m^2}=\frac{m^2(1+m^2)}{(1+m^2)^2}-\frac{m^2}{1+m^2}=0.$$

(5) (5)

(5) Hence Q lies on S.



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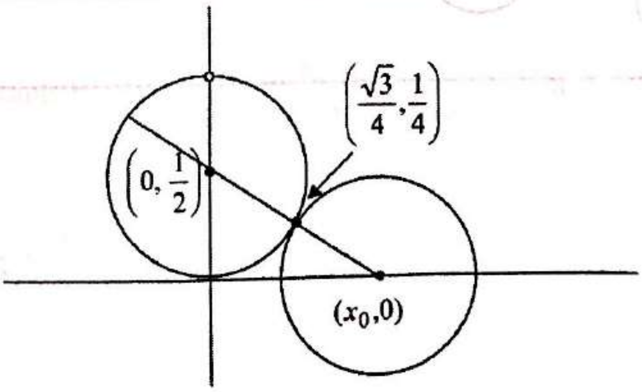
Put $x=\frac{\sqrt{3}}{4}$ and $y=\frac{1}{4}$ in x^2+y^2-y: (5)

$$x^2+y^2-y=\frac{3}{16}+\frac{1}{16}-\frac{1}{4}=0.$$ (5)

Hence, $\left(\frac{\sqrt{3}}{4},\frac{1}{4}\right)$ lies on S. (5)

15



Let x_0 be the x – coordinate of the centre of S'. Then

$$\sqrt{x_0^2+\frac{1}{4}}=\frac{1}{2}+\sqrt{\left(\frac{\sqrt{3}}{4}-x_0\right)^2+\frac{1}{16}}.$$ (5)

(5) (5)

$$\Rightarrow x_0^2+\frac{1}{4}=\frac{1}{4}+\sqrt{\left(\frac{\sqrt{3}}{4}-x_0\right)^2+\frac{1}{16}}.+\left(\frac{\sqrt{3}}{4}-x_0\right)^2+\frac{1}{16}.$$ (5)

$$\Rightarrow x_0=\frac{\sqrt{3}}{2}.$$ (5)

Hence the equation of S' is $\left(x-\frac{\sqrt{3}}{2}\right)^2+y^2=\left(\frac{\sqrt{3}}{4}\right)^2+\left(\frac{1}{4}\right)^2.$ (5)

i.e. $\left(x-\frac{\sqrt{3}}{2}\right)^2+y^2=\left(\frac{1}{2}\right)^2.$

30

The equation of the required circle touching S internally is

$$\left(x-\frac{\sqrt{3}}{2}\right)^2+y^2=\left(\frac{3}{2}\right)^2.$$ (10)

10

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17. (a) (i) Show that $\dfrac{2\cos(60° - \theta) - \cos\theta}{\sin\theta} = \sqrt{3}$ for $0° < \theta < 90°$.

(ii) In the quadrilateral $ABCD$ shown in the figure, $AB = AD$, $A\hat{B}C = 80°$, $C\hat{A}D = 20°$ and $B\hat{A}C = 60°$.

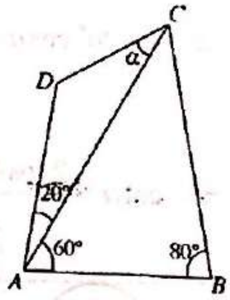
Let $A\hat{C}D = \alpha$. Using the Sine Rule for the triangle ABC, show that $\dfrac{AC}{AB} = 2\cos 40°$.

Next, using the Sine Rule for triangle ADC, show that $\dfrac{AC}{AD} = \dfrac{\sin(20° + \alpha)}{\sin\alpha}$.

Deduce that $\sin(20° + \alpha) = 2\cos 40° \sin\alpha$.

Hence, show that $\cot\alpha = \dfrac{2\cos 40° - \cos 20°}{\sin 20°}$.

Now, using the result in (i) above, show that $\alpha = 30°$.



(b) Solve the equation $\cos 4x + \sin 4x = \cos 2x + \sin 2x$.

(a) (i) $\dfrac{2\left\{\dfrac{1}{2}\cos\theta + \dfrac{\sqrt{3}}{2}\sin\theta\right\} - \cos\theta}{\sin\theta} = \sqrt{3}$. (5) (5) (5)

15

(ii) Using the Sine Rule: $\dfrac{AC}{\sin 80°} = \dfrac{AB}{\sin 40°}$. (10)

$\Rightarrow \dfrac{AC}{AB} = \dfrac{2\sin 40° \cos 40°}{\sin 40°} = 2\cos 40°$ (5) (5) (5)

Again, using the Sine Rule: $\dfrac{AC}{\sin(\alpha + 20°)} = \dfrac{AD}{\sin\alpha}$. (10)

$\Rightarrow \dfrac{AC}{AD} = \dfrac{\sin(20° + \alpha)}{\sin\alpha}$ (5)

Hence, $AB = AD \Rightarrow \dfrac{\sin(20^\circ + \alpha)}{\sin\alpha} = 2\cos 40^\circ.$ (5)

(5)

$\therefore \ \sin(20^\circ + \alpha) = 2\sin\alpha\cos 40^\circ$

$\Rightarrow \ \sin 20^\circ \cos\alpha + \cos 20^\circ \sin\alpha = 2\sin\alpha\cos 40^\circ$ (5)

$\Rightarrow \ \cot\alpha = \dfrac{2\cos 40^\circ - \cos 20^\circ}{\sin 20^\circ}$ (5) **60**

---

(10)

(i) with $\theta = 20^\circ \Rightarrow \dfrac{2\cos 40^\circ - \cos 20^\circ}{\sin 20^\circ} = \sqrt{3}$ (5)

$\therefore \cot\alpha = \sqrt{3}$ (5)

(5)

$\Rightarrow \alpha = 30^\circ$. (Since $0^\circ < \alpha < 90^\circ$) **25**

(b) $\cos 4x + \sin 4x = \cos 2x + \sin 2x$

$\Leftrightarrow \sin 4x - \sin 2x = \cos 2x - \cos 4x$ (5)

$\Leftrightarrow 2\cos 3x \sin x = 2\sin 3x \sin x$

(5) (5)

$\Leftrightarrow 2\sin x(\cos 3x - \sin 3x) = 0$ (5)

(5)

$\Leftrightarrow \sin x = 0$ or $\cos 3x = \sin 3x$

(5)

(5) (5)

$\Leftrightarrow \sin x = 0$ or $\tan 3x = 1$ $(\because \cos 3x \neq 0)$

$\Leftrightarrow x = n\pi$ for $n \in \mathbb{Z}$ or $3x = m\pi + \dfrac{\pi}{4}$ for $m \in \mathbb{Z}$ (5)

$\Leftrightarrow x = n\pi$ for $n \in \mathbb{Z}$ or $x = \dfrac{m\pi}{3} + \dfrac{\pi}{12}$ for $m \in \mathbb{Z}$ (5)

50

Confidential

Department of Examinations - Sri Lanka

G.C.E. (A/L) Examination - 2017

10 - Combined Mathematics II

Marking Scheme

This has been prepared for the use of marking examiners. Changes would be made according to the views presented at the Chief/Assistant Examiners' meeting.

Amendments to be included.

G.C.E. (A/L) Examination - 2017

10 - Combined Mathematics

Distribution of Marks

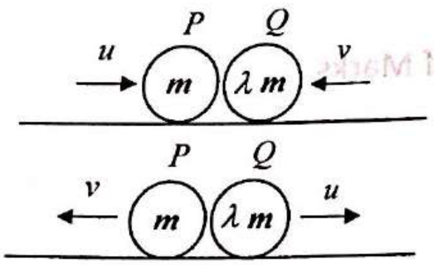
Paper I I:

Part A :	10 x 25 = 250	
Part B :	05 x 150 = 750	

Total = **1000/10**

Paper 11- Final Mark = **100**

1. A particle P of mass m and a particle Q of mass λm move with speeds u and v respectively, towards each other along the same straight line on a smooth horizontal floor. After their impact, the particle P moves with speed v and the particle Q moves with speed u in opposite directions. Show that $\lambda = 1$, and find the coefficient of restitution between P and Q.



Apply $\underline{I} = \Delta(M\underline{v})$ $\longrightarrow$ for the system:

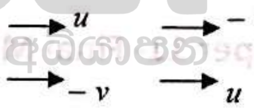
$$0 = (\lambda mu - mv) - (mu - \lambda mv)$$ (5)

$$\Rightarrow 0 = (\lambda - 1)u + (\lambda - 1)v$$

$$\Rightarrow 0 = (\lambda - 1)(u + v)$$

$$\Rightarrow \lambda = 1.$$ (5)

Let e be the coefficient of restitution between P and Q. Then, by Newton's law of restitution:

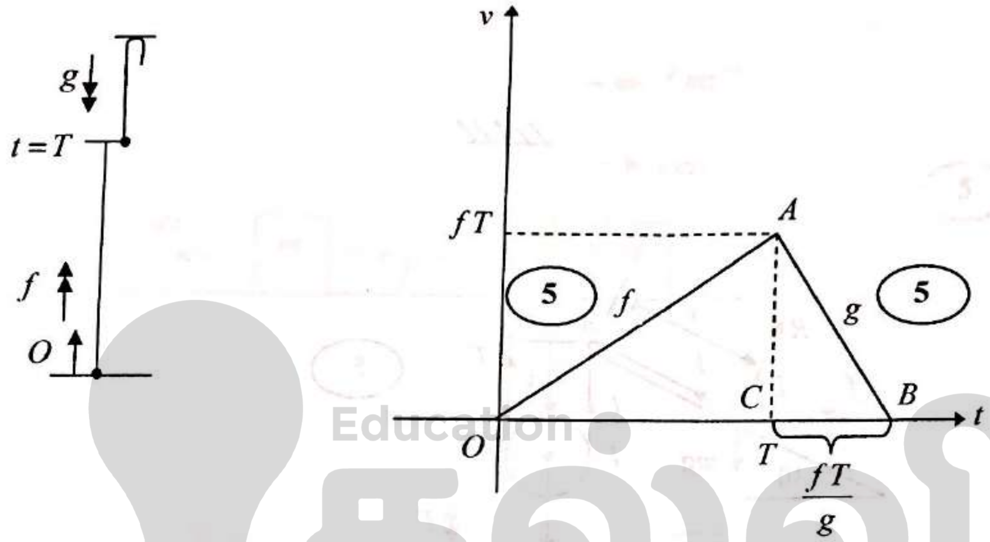


$$(u + v) = e(u + v)$$ (10)

$$\therefore \quad e = 1.$$ (5)

25

2. A balloon, carrying a small uniform ball, starts from rest from a point on the ground at time $t=0$ and moves vertically upwards with uniform acceleration f, where $f<g$. At time $t=T$, the ball gets gently detached from the balloon and moves under gravity. Sketch the velocity – time graph for the upward motion of the ball, from $t=0$ until the ball reaches its maximum height. Find the maximum height reached by the ball in terms of T, f and g.



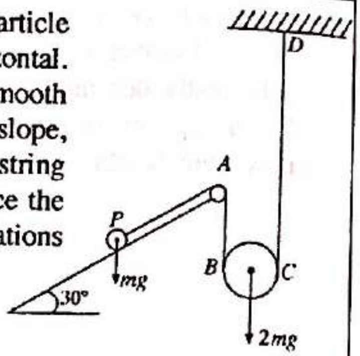
$f = \frac{AC}{T}$ and $g = \frac{AC}{BC} \Rightarrow BC = \frac{f}{g}T.$ (5)

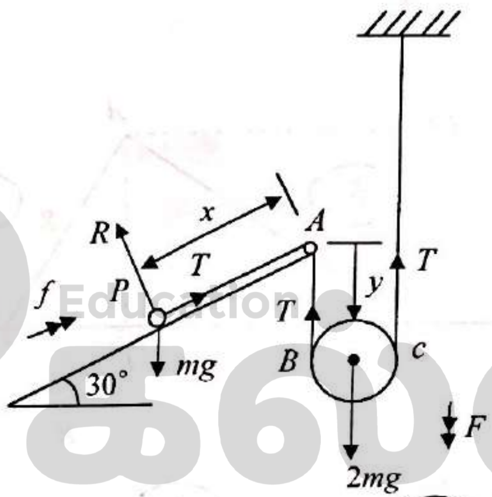
The maximum height = The area of the triangle $OAB = \frac{1}{2}\left(T + \frac{fT}{g}\right) \times fT.$ (5) + (5)

$$= \frac{fT^2}{2g}(f+g)$$

25

3. In the diagram, $PABCD$ is a light inextensible string attached to a particle of mass m placed on a fixed smooth plane inclined at $30°$ to the horizontal. The string passes over a fixed small smooth pulley at A and under a smooth pulley of mass $2m$. The point D is fixed. PA is along a line of greatest slope, and AB and CD are vertical. The system is released from rest with the string taut. Show that the magnitude of the acceleration of the particle is twice the magnitude of the acceleration of the movable pulley and write down equations sufficient to determine the tension of the string.





(5)

$x + 2y = \text{constant} \Rightarrow \ddot{x} + 2\ddot{y} = 0 \Rightarrow 2\ddot{y} = -\ddot{x}.$ (5)

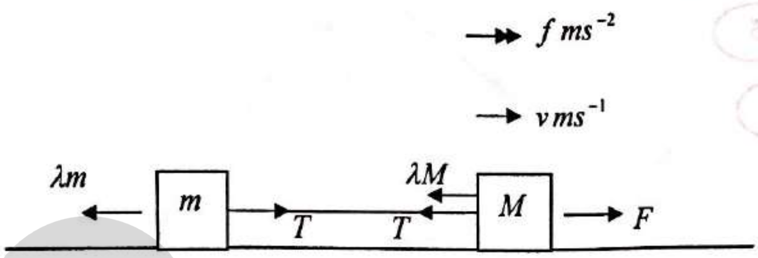
With f and F as shown in the figure, $f = 2F$. (5)

$\underline{F} = m\underline{a}$ ↗ for P: $T - mg \sin 30° = m f$ (5)

$\underline{F} = m\underline{a}$ ↓ for $2mg$: $2mg - 2T = 2m F$ (5)

25

4. A truck of mass M kg is towing a car of mass m kg, along a straight horizontal road, by means of a light inextensible cable which is parallel to the direction of motion of the truck and the car. The resistances to the motion of the truck and the car are λM newtons and λm newtons respectively, where $\lambda (>0)$ is a constant. At a certain instant, the power generated by the engine of the truck is P kW and the speed of the truck and the car is v m s^{-1}. Show that the tension of the cable at that instant is $\dfrac{1000mP}{(M+m)v}$ newtons.



(5)

Tractive force $F = \dfrac{1000P}{v}\,N$ ------------ (1) (5)

$\underline{F} = m\underline{a} \rightarrow$ for M : $F - \lambda M - T = M f$ ------------ (2) (5)

$\underline{F} = m\underline{a} \rightarrow$ for m : $T - \lambda m = m f$ ------------ (3) (5)

Now (1), (2) and (3) $\Rightarrow \dfrac{1000P}{v} - \lambda M - T = M f$

$\Rightarrow \dfrac{1000P}{v} - \lambda M - T = \dfrac{M}{m}(T - \lambda m)$

$\Rightarrow T = \dfrac{1000mP}{(M+m)v}\,N.$ (5)

25

5. In the usual notation, let $-\mathbf{i}+2\mathbf{j}$ and $2\alpha\mathbf{i}+\alpha\mathbf{j}$ be the position vectors of two points A and B respectively, with respect to a fixed origin O, where $\alpha(>0)$ is a constant. Using scalar product, show that $A\hat{O}B=\frac{\pi}{2}$.
Let C be the point such that $OACB$ is a rectangle. If the vector $\overrightarrow{OC}$ lies along the y-axis, find the value of α.

Scalar Product: $\overrightarrow{OA}\cdot\overrightarrow{OB}=(-\underline{i}+2\underline{j})\cdot(2\alpha\,\underline{i}+\alpha\,\underline{j})$

$=-2\alpha+2\alpha=0$ (5)

$\therefore A\hat{O}B=\frac{\pi}{2}.$ (5)

$\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}$

$=\overrightarrow{OA}+\overrightarrow{OB}$

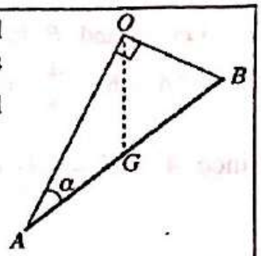
$=(-1+2\underline{\alpha})\underline{i}+(2+\alpha)\underline{j}$ (5)

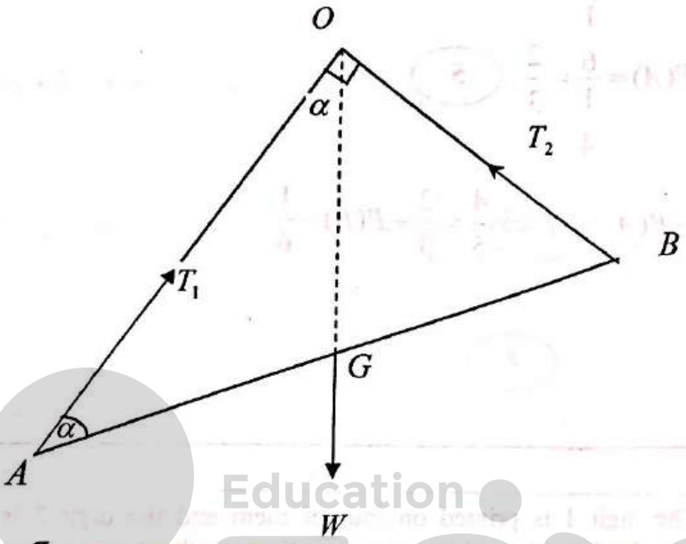
$\overrightarrow{OC}$ lies along the $y-axis \Rightarrow (1-2\alpha)=0$ (5)

$\Rightarrow \alpha=\frac{1}{2}.$ (5)

25

6. A uniform rod AB of length $2a$ and weight W, suspended from a fixed point O by two light inextensible strings OA and OB, is in equilibrium as shown in the figure. G is the mid-point of AB. It is given that $A\hat{O}B = \frac{\pi}{2}$ and $O\hat{A}B = \alpha$. Show that $A\hat{O}G = \alpha$, and find the tensions in the two strings.





(5)

Since $A\hat{O}B = \frac{\pi}{2}$, AB is a diameter of the circle through A, O and B, and G is the centre of it.

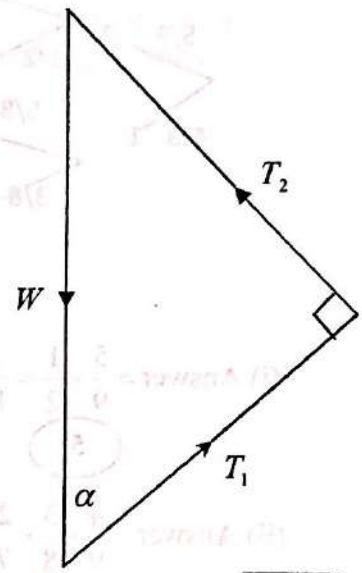
$\therefore AG = OG$.

$\Rightarrow A\hat{O}G = O\hat{A}G = \alpha$. (5)

Resolving along AO and BO (5)

$T_1 = W\cos\alpha$. (5)

$T_2 = W\sin\alpha$. (5)



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7. Let A and B be two events of a sample space Ω. In the usual notation, it is given that $P(A \cup B) = \frac{4}{5}$, $P(A' \cup B') = \frac{5}{6}$ and $P(B \mid A) = \frac{1}{4}$. Find $P(A)$ and $P(B)$.

Since $A' \cup B' = (A \cap B)'$, we have $P((A \cap B)') = 1 - P(A \cap B)$.

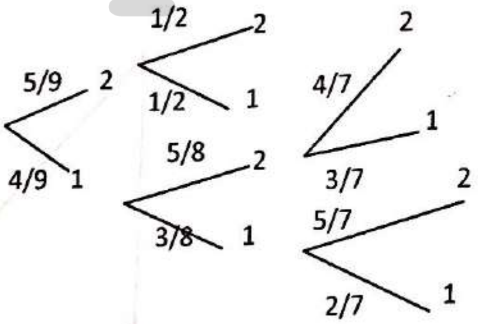
$$\therefore P(A \cap B) = 1 - \frac{5}{6} = \frac{1}{6}.$$ (5)

Now $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)}$ (5) $\Rightarrow P(A) = \dfrac{\frac{1}{6}}{\frac{1}{4}} = \dfrac{2}{3}.$ (5)

Also, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (5) $\Rightarrow \frac{4}{5} = \frac{2}{3} + P(B) - \frac{1}{6}$

$$\Rightarrow P(B) = \frac{4}{5} - \frac{1}{2} = \frac{3}{10}.$$ (5)

8. A bag contains nine cards. The digit 1 is printed on four of them and the digit 2 is printed on the rest. Cards are drawn from the bag at random, one at a time, without replacement. Find the probability that
(i) the sum of the digits on the first two cards drawn is four,
(ii) the sum of the digits on the first three cards drawn is three.



(i) Answer $= \frac{5}{9} \times \frac{1}{2} = \frac{5}{18}.$ (5) (5)

(ii) Answer $= \frac{4}{9} \times \frac{3}{8} \times \frac{2}{7} = \frac{1}{21}.$ (5) (10)

25

9. Values of six observations are a, a, b, b, x and y, where a, b, x and y are distinct positive integers with $a<b$. What are the modes of these six observations?
It is given that the sum and the product of these modes are x and y respectively. If the mean of the six observations is $\frac{7}{2}$, find a and b.

Modes are a and b. (5)

It is given that $a+b=x$ and $ab=y$.

Since the mean is $\frac{7}{2}$, we have $\frac{2a+2b+x+y}{6}=\frac{7}{2}$. (5)

$\therefore 3a+3b+ab=21$ ---------------- (1) (5)

(1) $\Rightarrow ab$ is divisible by 3 and hence $ab \geq 3$.

Again, (1) $\Rightarrow a+b \leq 6$. (5)

Since $1<a<b$, we must have

$a=2 \quad b=3$ (5)

25

10. The mean and the variance of the ten numbers $x_1, x_2, \ldots, x_{10}$ are 10 and 9 respectively. It is given that the mean of the nine numbers which remain after omitting the number x_{10} is also 10. Find the variance of these nine numbers.

$$\text{Mean}=10 \Rightarrow \frac{\sum_{i=1}^{10} x_i}{10}=10.$$ (5)

$$\text{Variance}=9 \Rightarrow \frac{\sum_{i=1}^{10} x_i^2}{10}-10^2=9 \Rightarrow \sum_{i=1}^{10} x_i^2=1090.$$ (5)

The mean of the first nine numbers $=10 \Rightarrow x_{10}=10$. (5)

$$\therefore \sum_{i=1}^{9} x_i^2=990.$$ (5)

$\therefore$ The variance of the first nine numbers $=\frac{990}{9}-10^2=10$. (5)

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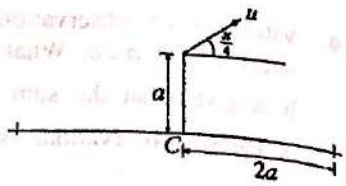
11. (*a*) The base of a vertical tower of height a is at the centre C of a circular pond of radius $2a$, on horizontal ground. A small stone is projected from the top of the tower with speed u at an angle $\frac{\pi}{4}$ above the horizontal. (See the figure.) The stone moves freely under gravity and hits the horizontal plane through C at a distance R from C. Show that R is given by the equation $gR^2 - u^2R - u^2a = 0$:

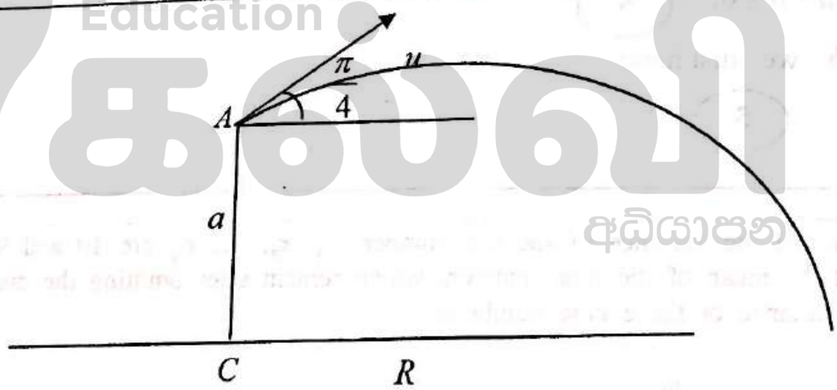


Find R in terms of u, a and g, and deduce that if $u^2 > \frac{4}{3}ga$, then the stone will not fall into the pond.

(*b*) A ship S is sailing due East with uniform speed u km h^{-1}, relative to earth. At the instant when the ship is at a distance l km at an angle θ South of West from a boat B, the boat travels in a straight line path, intending to intercept the ship, with uniform speed v km h^{-1} relative to earth, where $u\sin\theta < v < u$. Assuming that the ship and the boat maintain their speeds and paths, sketch, in the same diagram, the velocity triangles to determine the two possible paths of the boat relative to earth.

Show that the angle between the two possible directions of motion of the boat relative to earth is $\pi - 2\alpha$, where $\alpha = \sin^{-1}\left(\frac{u\sin\theta}{v}\right)$.

Let t_1 hours and t_2 hours be the times taken by the boat to intercept the ship along these two paths. Show that $t_1 + t_2 = \frac{2lu\cos\theta}{u^2 - v^2}$.



Apply $s = ut + \frac{1}{2}at^2$,

$\rightarrow$ from A to B: $R = u\cos\frac{\pi}{4}\cdot t = \frac{ut}{\sqrt{2}}$ -------------- (1) **5**

$\uparrow$ from A to B $-a = u\sin\frac{\pi}{4}t - \frac{1}{2}gt^2$ --------------(2) **10**

(1) and (2) $\Rightarrow$ $-a = R - \frac{1}{2}g\frac{2R^2}{u^2}$ **5**

$\Rightarrow gR^2 - u^2R - u^2a = 0$ **5**

25

$\therefore R = \dfrac{u^2 \pm \sqrt{u^4 + 4u^2ag}}{2g}$. (5)

$R = \dfrac{u^2 + \sqrt{u^4 + 4agu^2}}{2g}$ (5) $(\because R > 0)$ (5)

15

Suppose that $u^2 > \dfrac{4}{3}ga$.

Then, $R > \dfrac{\frac{4}{3}ga + \sqrt{\frac{16}{9}g^2a^2 + \frac{16}{3}g^2a^2}}{2g} = \dfrac{\frac{4}{3}ga + \frac{8}{3}ga}{2g} = 2a.$ (5) (5)

$\Rightarrow R > 2a$.

$\therefore$ The stone will not fall into the pond.

10

(b) $\underline{V}(S, E) = \rightarrow u$ (5)

$\underline{V}(B, E) = v$ (5)

$\underline{V}(B, S) =$  (5)

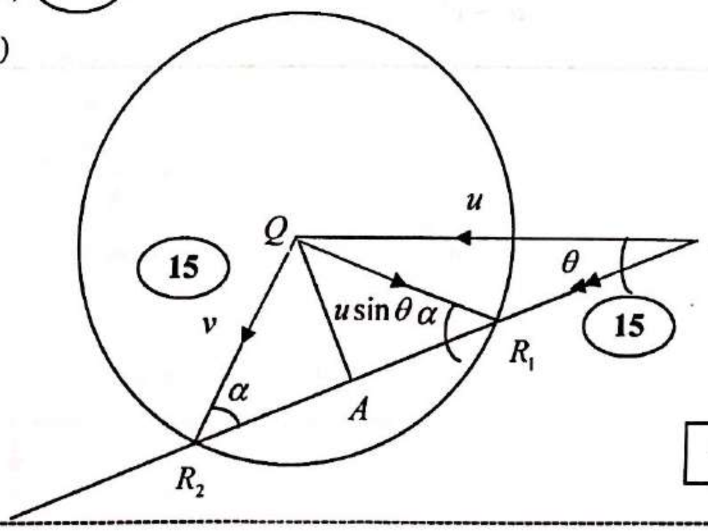
$u\sin\theta < v < u$

$\underline{V}(B, S) = \underline{V}(B, E) + \underline{V}(E, S)$ (5)

$= \underline{V}(E, S) + \underline{V}(B, E)$

$= \overrightarrow{PQ} + \overrightarrow{QR}$

$= \overrightarrow{PR}$.



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Required angle $= R_1\hat{Q}R_2$ (5)

$= \pi - 2\alpha$, where $Q\hat{R}_2R_1 = \alpha$. (5)

$$\sin\alpha = \frac{QA}{QR_2} = \frac{u\sin\theta}{v}$$ (5)

$$\therefore \alpha = \sin^{-1}\left(\frac{u\sin\theta}{v}\right).$$

15

$$t_1 + t_2 = \frac{l}{PR_1} + \frac{l}{PR_2} = \frac{l(PR_1 + PR_2)}{PR_1 \cdot PR_2}.$$

(5)

$$PR_1 = PA - AR_1$$
$$= u\cos\theta - \sqrt{v^2 - u^2\sin^2\theta}$$ (10)

$$PR_2 = PA + AR_2$$
$$= u\cos\theta + \sqrt{v^2 - u^2\sin^2\theta}$$ (10)

$$\therefore t_1 + t_2 = \frac{l\ 2u\cos\theta}{u^2\cos^2\theta - (v^2 - u^2\sin^2\theta)}$$ (5)

$$= \frac{2lu\cos\theta}{u^2 - v^2}\ (\because \cos^2\theta + \sin^2\theta = 1)$$ (5)

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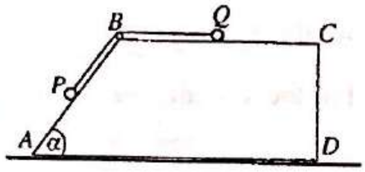
12. (*a*) The trapezium $ABCD$ shown in the figure is a vertical cross-section through the centre of gravity of a smooth uniform block of mass $2m$. The lines AD and BC are parallel, and the line AB is a line of greatest slope of the face containing it. Also, $AB = 2a$ and $B\hat{A}D = \alpha$, where $0 < \alpha < \frac{\pi}{2}$ and $\cos\alpha = \frac{3}{5}$. The block is placed with the face containing AD on a smooth horizontal floor.

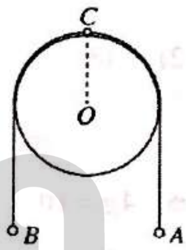


A light inextensible string of length l ($> 2a$) passes over a small smooth pulley at B, and has a particle P of mass m attached to one end and another particle Q of the same mass m attached to the other end. The system is released from rest with the string taut, the particle P held at the mid-point of AB and the particle Q on BC, as shown in the figure.

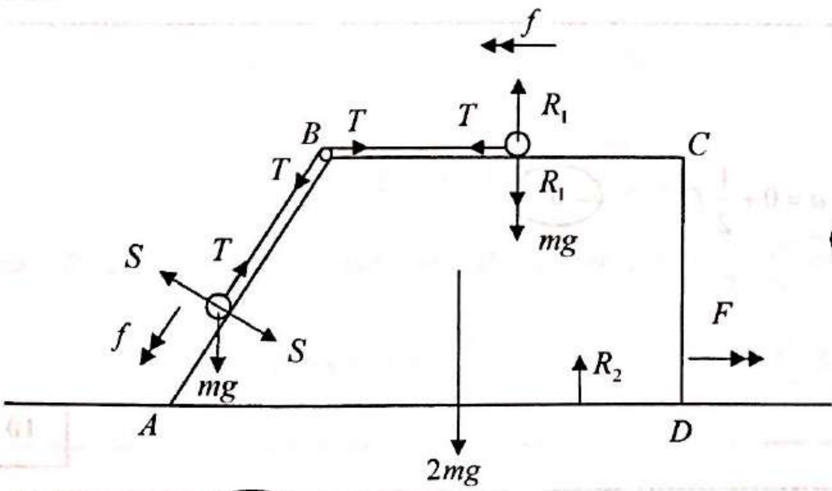
Show that the acceleration of the block relative to the floor is $\frac{4}{17}g$ and find the acceleration of P relative to the block.

Also, show that the time taken by the particle P to reach A is $\sqrt{\frac{17a}{5g}}$.

(*b*) Two particles A and B, each of mass m are attached to the two ends of a light inextensible string of length $l (> 2\pi a)$. A particle C of mass $2m$ is attached to the mid-point of the string. The string is placed over a fixed smooth sphere of centre O and radius a with the particle C at the highest point of the sphere, and the particles A and B hanging freely in a vertical plane through O, as shown in the figure. The particle C is given a small displacement on the sphere in the same vertical plane, so that the particle A moves downwards in a straight line path. As long as the particle C is in contact with the sphere, show that $\dot{\theta}^2 = \frac{g}{a}(1-\cos\theta)$, where θ is the angle through which OC has turned. Show further that the particle C leaves the sphere when $\theta = \frac{\pi}{3}$.



(a)



(10)

(5)

Let $\underline{a}(P, \text{Block}) = f \swarrow$ Then $\underline{a}(Q, \text{Block}) = f \twoheadleftarrow$

Also, let $\underline{f}(\text{Block}, E) = F \twoheadrightarrow$

Apply $\underline{F} = m\underline{a}$:

For the system $\rightarrow$ $0 = 2mF + m(F - f) + m(F - f\cos\alpha)$ (10)

$\Rightarrow 0 = 4F - f - f \times \frac{3}{5}$

$\therefore f = \frac{5F}{2}$ ------------(1) (5)

For the particle P ↙ $mg\sin\alpha - T = m(f - F\cos\alpha)$ ------------ (2) (10)

For the particle Q ← $T = m(f - F)$ ------------(3) (10)

(5)

(2) + (3) $\Rightarrow$ $mg \times \frac{4}{5} = m(f - F) + m\left(f - F \times \frac{3}{5}\right)$

$\Rightarrow 4g = 5f - 5F + 5f - 3F$

$\Rightarrow 4g = 10f - 8F$ (5)

Now (1) $\Rightarrow$ $4g = 25F - 8F$

$\Rightarrow F = \frac{4}{17}g$. (5)

(1) $\Rightarrow f = \frac{10g}{17}$ (5)

70

Apply $s = ut + \frac{1}{2}at^2$: ↙

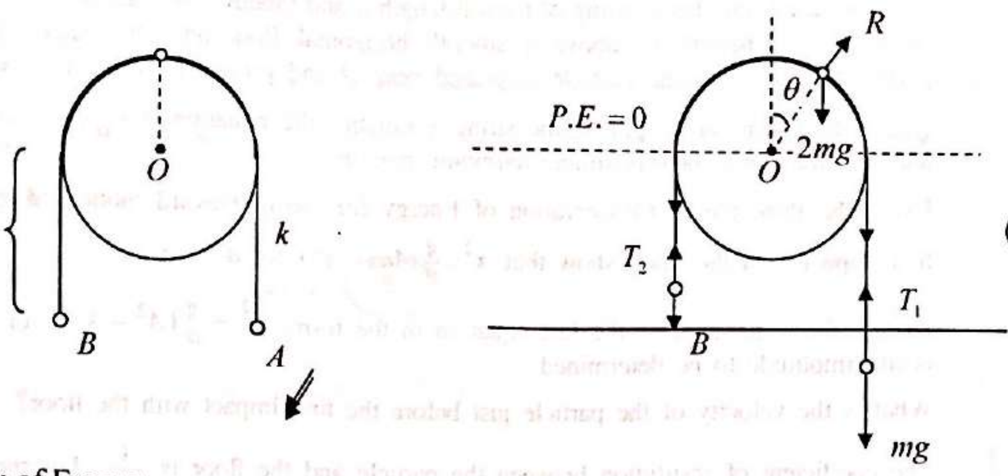
For the motion of (P, B): $a = 0 + \frac{1}{2}ft^2$ (5)

(5)

$\therefore t = \sqrt{\frac{2a}{\frac{10g}{17}}} = \sqrt{\frac{17a}{5g}}$.

10

(b)



Conservation of Energy:

$$\frac{1}{2}\times 2m\times(a\dot{\theta})^2 + 2\times\frac{1}{2}\times m\times(a\dot{\theta})^2 + 2mga\cos\theta - mg(k-a\theta) - mg(k+a\theta) = -2mgk + 2mga$$

(25) PE 10, KE 10, Equation 5

$\Rightarrow 2a\dot{\theta}^2 = -2g\cos\theta + 2g$ (10)

$\therefore \theta^2 = \frac{g}{a}(1-\cos\theta)$.

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Apply $\underline{F} = m\underline{a}$:

For C ↗ ; $R - 2mg\cos\theta = -2m\cdot a\dot{\theta}^2$ (10)

$\Rightarrow R = 2mg\cos\theta - 2mg(1-\cos\theta)$

$= 2mg(2\cos\theta - 1)$. (5)

$\Rightarrow$ R decreases as θ increases, and $R = 0$ when $\cos\theta = \frac{1}{2}$. (5)

$\therefore$ C leaves the sphere when $\theta = \frac{\pi}{3}$. (5)

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13. One end of a light elastic string of natural length a and modulus of elasticity mg is attached to a fixed point O at a height $3a$ above a smooth horizontal floor and the other end is attached to a particle of mass m. The particle is placed near O and projected vertically downwards with speed $\sqrt{ga}$. Show that the length of the string x satisfies the equation $\ddot{x}+\frac{g}{a}(x-2a)=0$ for $a \le x < 3a$, and find the centre of this simple harmonic motion.

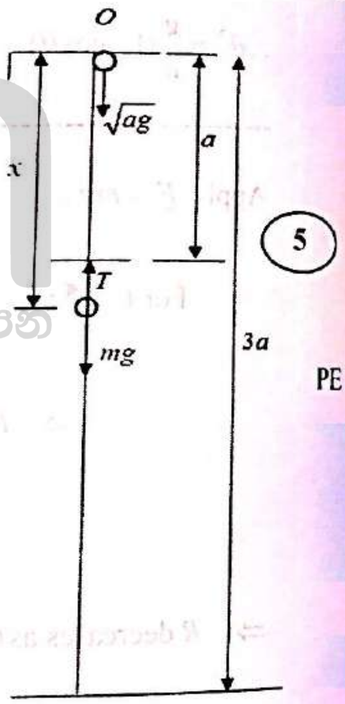
Using the Principle of Conservation of Energy for the downward motion of the particle until the first impact with the floor, show that $\dot{x}^2=\frac{g}{a}(4ax-x^2)$ for $a \le x < 3a$.

Taking $X=x-2a$, express the last equation in the form $\dot{X}^2=\frac{g}{a}\left(A^2-X^2\right)$ for $-a \le X < a$, where A is the amplitude to be determined.

What is the velocity of the particle just before the first impact with the floor?

The coefficient of restitution between the particle and the floor is $\frac{1}{\sqrt{3}}$. For the upward motion of the particle after the first impact, until the string becomes slack, it is given that $\dot{X}^2=\frac{g}{a}\left(B^2-X^2\right)$ for $-a \le X < a$, where B is the amplitude of this new simple harmonic motion to be determined.

Show that the total time during which the particle performs downwards and upwards simple harmonic motions described above is $\frac{5\pi}{6}\sqrt{\frac{a}{g}}$.



For $a \le x < 3a$:

$$T=\frac{mg}{a}(x-a) \quad (5)$$

Apply $\underline{F}=m\underline{a}$:

For $m \downarrow$; $mg-T=m\ddot{x}$ (10)

$$\Rightarrow mg-\frac{mg}{a}(x-a)=m\ddot{x}$$

$$\Rightarrow \ddot{x}+\frac{g}{a}(x-2a)=0. \quad (5)$$

The centre is given by $\ddot{x}=0$ .i.e. $x=2a$.
(5)

So the centre is at the point C, where C is vertically below O with $OC=2a$. (5)

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Conservation of Energy : $\frac{1}{2}m(ga)=\frac{1}{2}m\dot{x}^2-mgx+\frac{1}{2}mg\frac{(x-a)^2}{a}$ (20)

$$ga=\dot{x}^2-2gx+\frac{g}{a}(x^2-2ax+a^2)$$

$$\dot{x}^2=2gx-\frac{g}{a}x^2+2gx$$

$\dot{x}^2 = \frac{g}{a}(4ax - x^2)$ for $a \le x < 3a$ (5)

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$X = x - 2a \Rightarrow \dot{X} = \dot{x}$ (5)

also $a \le x < 3a \Leftrightarrow -a \le X < a.$

$\dot{X}^2 = \frac{g}{a}\{4a(X+2a) - (X+2a)^2\}$ (5)

$= \frac{g}{a}\{4a^2 - X^2\}$ for $-a \le X < a$ (5)

$\therefore A = 2a.$ (5)

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Let $\downarrow v$ be the velocity of the particle just before impact.

Then $v^2 = \frac{g}{a}(4a^2 - a^2) = 3ga$ (5)

$\therefore v = \sqrt{3ga}.$ (5)

10

By Newton's law of restitution, velocity $\uparrow$ just after impact $= \sqrt{ga}\left(\because e = \frac{1}{\sqrt{3}}\right).$ (10)

$\dot{X}^2 = \frac{g}{a}(B^2 - X^2)$

When $X = a$

$ga = \frac{g}{a}(B^2 - a^2)$ (5)

$\Rightarrow B = \sqrt{2}a.$ (5)

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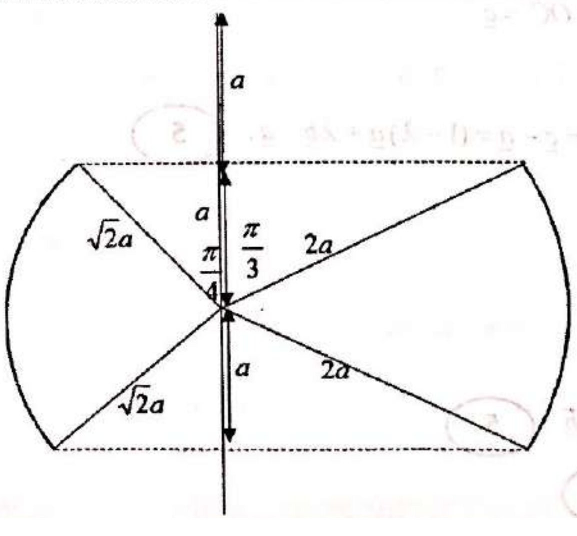

(10) (15)

Since $\sqrt{\frac{g}{a}}t_1 = \frac{\pi}{3}$, we have $t_1 = \frac{\pi}{3}\sqrt{\frac{a}{g}}$. (5)

Since $\sqrt{\frac{g}{a}}t_2 = \frac{\pi}{2}$, we have $t_2 = \frac{\pi}{2}\sqrt{\frac{a}{g}}$. (5)

$\therefore t_1 + t_2 = \frac{5\pi}{6}\sqrt{\frac{a}{g}}$. (5)

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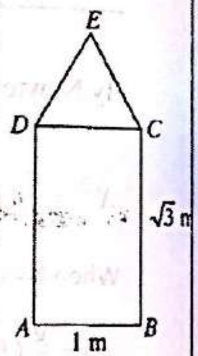
14. (a) The position vectors of two distinct points A and B with respect to a fixed origin O, not collinear with A and B, are $\mathbf{a}$ and $\mathbf{b}$ respectively. Let $\mathbf{c} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$ be the position vector of a point C with respect to O, where $0 < \lambda < 1$.
Express the vectors $\overrightarrow{AC}$ and $\overrightarrow{CB}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and λ.
Hence, show that the point C lies on the line segment AB and that $AC : CB = \lambda : (1-\lambda)$.
Now, suppose that the line OC bisects the angle AOB. Show that $|\mathbf{b}|(\mathbf{a}\cdot\mathbf{c}) = |\mathbf{a}|(\mathbf{b}\cdot\mathbf{c})$ and **hence**, find λ.

(b) In the figure, $ABCD$ is a rectangle with $AB = 1$ m and $BC = \sqrt{3}$ m, and CDE is an equilateral triangle. Forces of magnitude 5, $2\sqrt{3}$, 3, $4\sqrt{3}$, P and Q newtons act along BA, DA, DC, CB, CE and DE respectively, in the directions indicated by the order of the letters. This system of forces reduces to a couple. Show that $P = 4$ and $Q = 8$, and find the moment of this couple.
Now, the directions of forces acting along BA and DA are reversed, but their magnitudes remain the same. Show that the new system reduces to a single resultant force of magnitude $2\sqrt{37}$ newtons.
Show further that the distance from A to the point at which the line of action of this resultant force meets BA produced is $\frac{7}{4}$ m.



14. (a) $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$ and $\overrightarrow{OC} = \underline{c}$

(5)

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = \overrightarrow{OC} - \overrightarrow{OA} = \underline{c} - \underline{a} = (1-\lambda)\underline{a} + \lambda\underline{b} - \underline{a}$. (5)

$= \lambda(\underline{b} - \underline{a})$.

(5)

$\overrightarrow{CB} = \underline{b} - \underline{c} = \underline{b} - (1-\lambda)\underline{a} - \lambda\underline{b}$ (5)

$= (1-\lambda)(\underline{b} - \underline{a})$. (5)

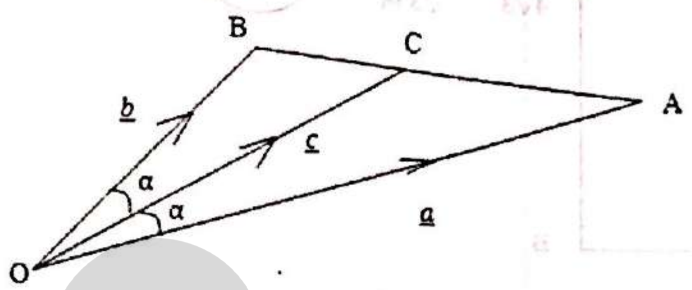
25

$\overrightarrow{AC} = \frac{\lambda}{(1-\lambda)}\overrightarrow{CB}$ (5)

(5)

$\therefore C$ lies on AB and $\frac{AC}{CB} = \frac{\lambda}{(1-\lambda)}$.

i.e. $AC : CB = \lambda : (1-\lambda)$ (5) **15**



$B\hat{O}C = A\hat{O}C$

$\underline{a} \cdot \underline{c} = |\underline{a}||\underline{c}|\cos\alpha$ (5)

$b \cdot \underline{c} = |b||\underline{c}|\cos\alpha$ (5)

$\Rightarrow \frac{\underline{a}\cdot\underline{c}}{|\underline{a}|} = \frac{\underline{b}\cdot\underline{c}}{|\underline{b}|}$ (5)

$\Rightarrow |\underline{b}|(\underline{a}\cdot\underline{c}) = |\underline{a}|(\underline{b}\cdot\underline{c})$ (5) **20**

---

(5)

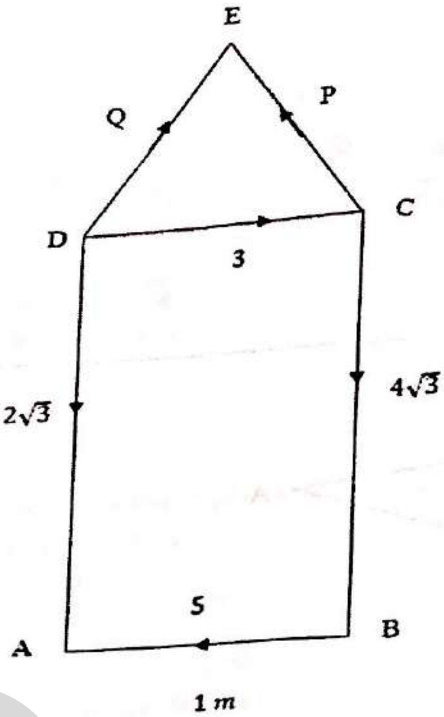
$\Rightarrow |\underline{b}|\{(1-\lambda)|\underline{a}|^2 + \lambda\,\underline{a}\cdot\underline{b}\} = |\underline{a}|\{(1-\lambda)\,\underline{a}\cdot\underline{b} + \lambda|\underline{b}|^2\}$ (5)

(5)

$(1-\lambda)|\underline{a}|\{|\underline{a}||\underline{b}| - a\cdot\underline{b}\} = \lambda|\underline{b}|\{|\underline{a}||\underline{b}| - \underline{a}\cdot\underline{b}\}$

$(1-\lambda)|\underline{a}| = \lambda|\underline{b}|$

$\Rightarrow \lambda = \frac{|\underline{a}|}{|\underline{a}|+|\underline{b}|}$. ($\because \underline{a}$ and $\underline{b}$ are distinct and non-collinear.) **15**



$\sqrt{3}\,m$ (10)

Since the system reduces to a couple,

$\rightarrow \quad 3-5+Q\cos 60^\circ - P\cos 60^\circ = 0$ (5)

$\Rightarrow \quad P-Q=-4,$ and (5)

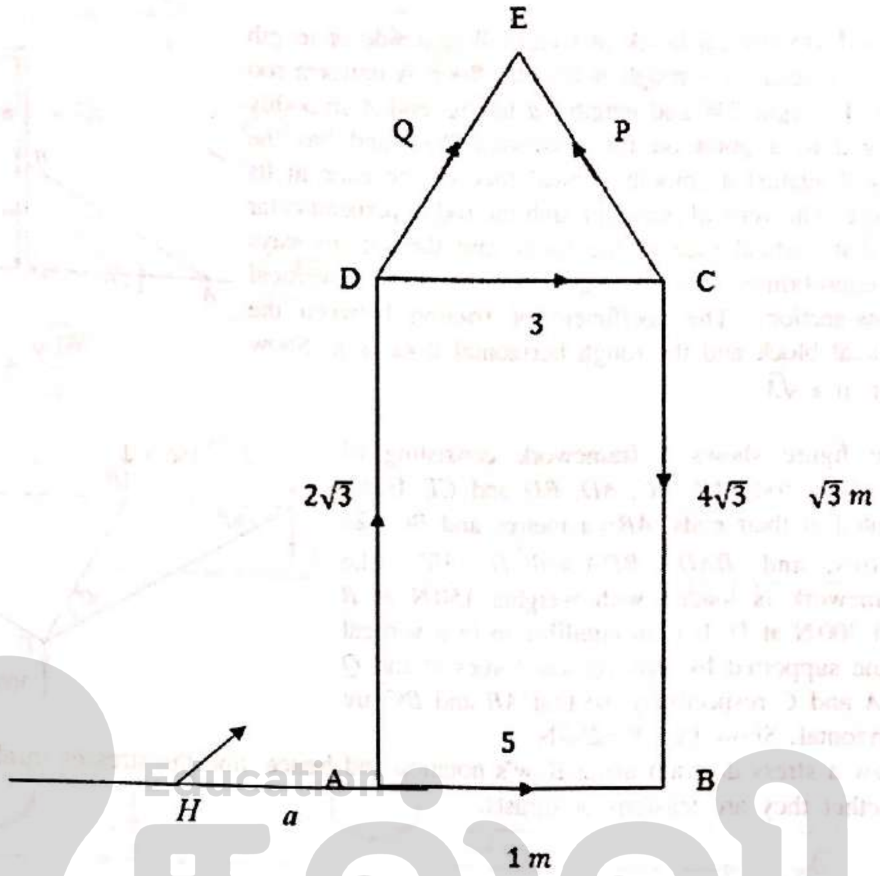
$\uparrow \quad -2\sqrt{3}-4\sqrt{3}+Q\sin 60^\circ + P\sin 60^\circ = 0$ (5)

$\Rightarrow \quad P+Q=12$ (5)

$\therefore \quad P=4$ and $Q=8.$ (5)

E↻ Moment of the couple = $7\sqrt{3}\,Nm$ (10)

45



$\rightarrow$ $X = 5 + 3 + 8\cos 60^\circ - 4\cos 60^\circ = 10$ (5)

$\uparrow$ $Y = 2\sqrt{3} - 4\sqrt{3} + 8\sin 60^\circ + 4\sin 60^\circ = 4\sqrt{3}$ (5)

$R = \sqrt{100 + 48} = 2\sqrt{37}$ (5)

15

Let H be the point at which on BA produced meets the line of action

$H \circlearrowleft$

$-6\sqrt{3} + 2\sqrt{3}(1 + a) + \sqrt{3}(3 + 4 - 2) = 0$ (10)

$-6a + 2 + 2a + 5 = 0$

$a = \frac{7}{4}m.$ (5)

15

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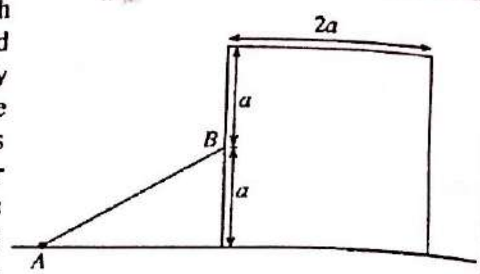
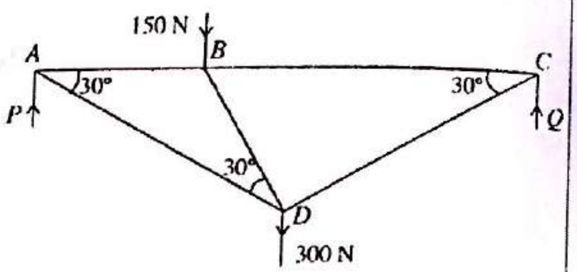
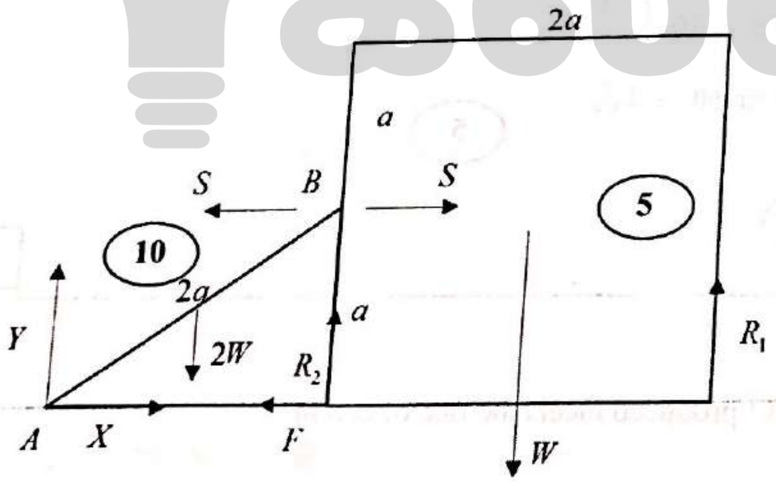
15. (*a*) A uniform cubical block of weight W and side of length $2a$ is placed on a rough horizontal floor. A uniform rod AB of weight $2W$ and length $2a$ has its end A smoothly hinged to a point on the horizontal floor and has the end B against a smooth vertical face of the cube at its centre. The vertical plane through the rod is perpendicular to that vertical face of the block and the system stays in equilibrium. (See the figure for the relevant vertical cross-section.) The coefficient of friction between the cubical block and the rough horizontal floor is μ. Show that $\mu \geq \sqrt{3}$.



(*b*) The figure shows a framework consisting of five light rods AB, BC, AD, BD and CD freely jointed at their ends. $AB = a$ metres and $BC = 2a$ metres, and $B\hat{A}D = B\hat{D}A = B\hat{C}D = 30°$. The framework is loaded with weights 150 N at B and 300 N at D. It is in equilibrium in a vertical plane supported by two vertical forces P and Q at A and C respectively, so that AB and BC are horizontal. Show that $P = 250$ N.



Draw a stress diagram using Bow's notation and **hence**, find the stresses in all the rods and state whether they are tensions or thrusts.



For the block ↑

$R_1 + R_2 = W$ (10)

For the block ⟶

$F = S$ (5)

For AB, $\overset{\curvearrowleft}{A}$ $S \times a - 2W \times \frac{\sqrt{3}a}{2} = 0$ (10)

$\therefore S = \sqrt{3}W$ (5)

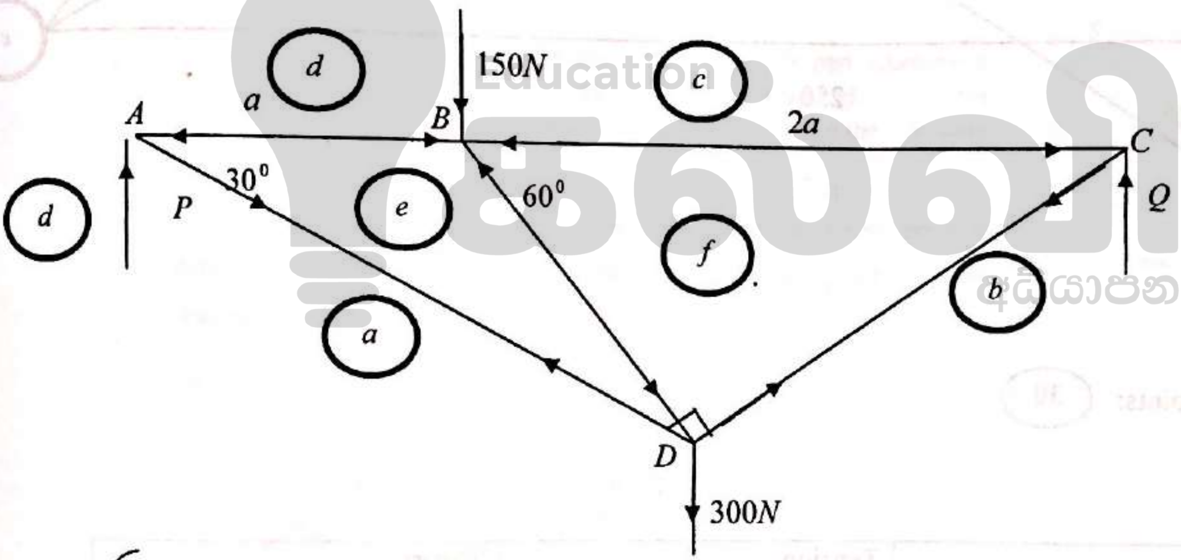
$\mu \geq \frac{|F|}{(R_1 + R_2)}$ (10)

$\mu \geq \frac{\sqrt{3}W}{W}$

$\mu \geq \sqrt{3}$ (5)

60

(b)

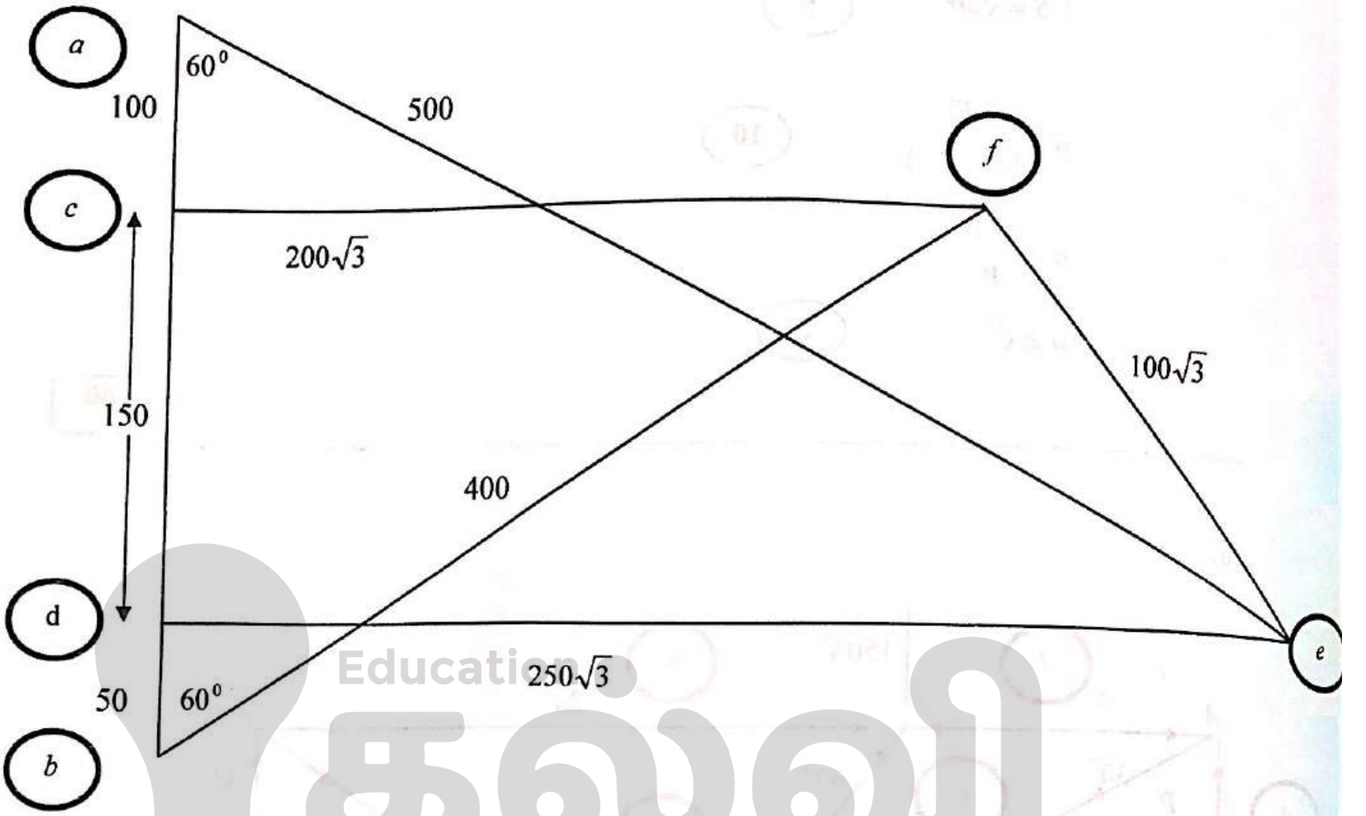


$\overset{\curvearrowright}{C}$

$150 \times 2a + 300\left(2a - \frac{a}{2}\right) - P.3a = 0$ (5)

$\Rightarrow P = 250N$ (5)

10



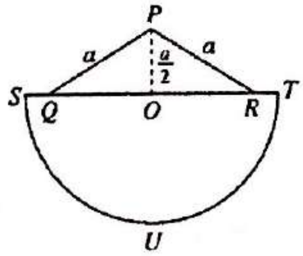
Three joints: (30)

Rod	Tension	Thrust
AB		$250\sqrt{3}\,N$ (10)
BC		$200\sqrt{3}\,N$ (10)
CD	$400\,N$ (10)	
DA	$500\sqrt{3}\,N$ (10)	
DB		$100\sqrt{3}\,N$ (10)

80

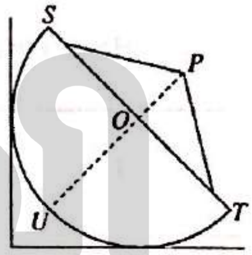
16. Show that the centre of mass of a thin uniform wire in the shape of a semi-circular arc of centre C and radius a, is at a distance $\frac{2a}{\pi}$ from C.

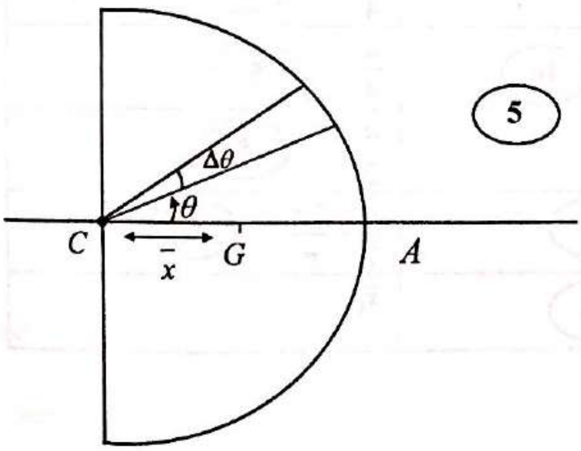
In the adjoining figure, PQ, PR and ST are three straight line pieces cut from a thin uniform wire of mass ρ per unit length. The two pieces PQ and PR are welded to each other at the point P and then welded to ST at the points Q and R. It is given that $PQ = PR = a$, $ST = 2a$ and $PO = \frac{a}{2}$, where O is the mid-point of both QR and ST. Also, SUT is a semicircular arc of centre O and radius a made up of a thin uniform wire of mass $k\rho$ per unit length, where $k(>0)$ is a constant. The rigid plane wire frame L shown in the figure has been made by welding the semicircular wire SUT to the wire ST in the plane of PQR at the points S and T. Show that the centre of mass of L is at a distance $\left(\frac{\pi k + 4k + 3}{\pi k + 4}\right)\frac{a}{2}$ from P.



The wire frame L is in equilibrium with its circular part touching a smooth vertical wall and a horizontal ground rough enough to prevent slipping, and its plane perpendicular to the wall as shown in the adjoining figure. Mark the forces acting on L and show that $k > \frac{1}{4}$.

Now, let $k = 1$. The equilibrium is maintained in the above position even after a particle of mass m is attached to L at the point P. Show that $m < 3\rho a$.





(5)

By symmetry, centre of mass, G lies on CA and OG=$\bar{x}$ (5)

Let ρ be the mass per unit length.

Then $\Delta m = a(\Delta\theta)\rho$ and $\bar{x} = \dfrac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\rho a\cos\theta \, d\theta}{\pi a \rho}$ (10) (5) $= \dfrac{a}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \, d\theta$

$$= \frac{a}{\pi}\cdot \sin\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \quad (5)$$

$$= \frac{a}{\pi}\left[2\sin\frac{\pi}{2}\right]$$

$$= \frac{2a}{\pi}. \quad (5)$$

Hence the centre of mass is at a distance $\dfrac{2a}{\pi}$ from C. **35**

Object	Mass	Vertical distance from P to the centre of Mass ↓
PR	$a\rho$	$\frac{a}{4}$ (5)
PQ	$a\rho$ (10)	$\frac{a}{4}$ (5)
ST	$2a\rho$	$\frac{a}{2}$ (5)
SUT	$\pi a k \rho$ (5)	$\frac{a}{2}+\frac{2a}{\pi}$ (5)
Combined Object	$(4+\pi k)a\rho$ (5)	$\bar{x}_1$

By symmetry, centre of mass of L lies on the line joining P and O. (5)

By the definition of centre of mass,

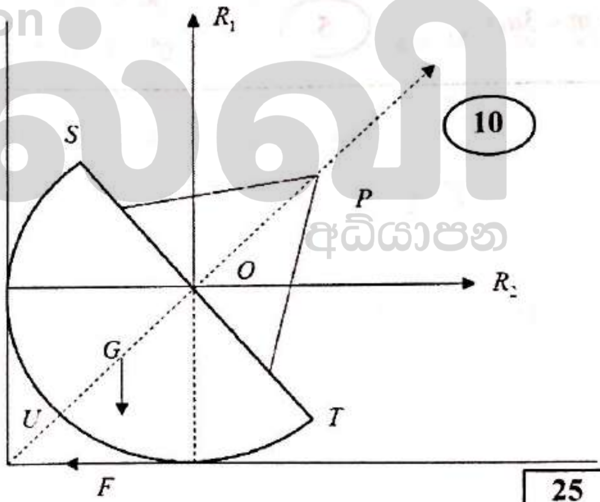
$$(4a\rho+\pi a k \rho)\overline{x_1}=2a\rho\times\frac{a}{4}+2a\rho\times\frac{a}{2}+\pi ak\rho\times\left(\frac{a}{2}+\frac{2a}{\pi}\right) \quad (15)$$

$$\Rightarrow (4+\pi k)\overline{x_1}=\frac{a}{2}+a+\frac{\pi a k}{2}+2ak \quad (5)$$

$$\Rightarrow \overline{x_1}=\left(\frac{\pi k+4k+3}{\pi k+4}\right)\frac{a}{2}. \quad (5)$$

70

For the wire frame L to be in equilibrium in the given position, we must have $\overline{x_1}>\frac{a}{2}$. (5)



(10)

i.e. $\left(\frac{\pi k+4k+3}{\pi k+4}\right)\frac{a}{2}>\frac{a}{2}$. (5)

$\Leftrightarrow \quad \pi k+4k+3>\pi k+4.$

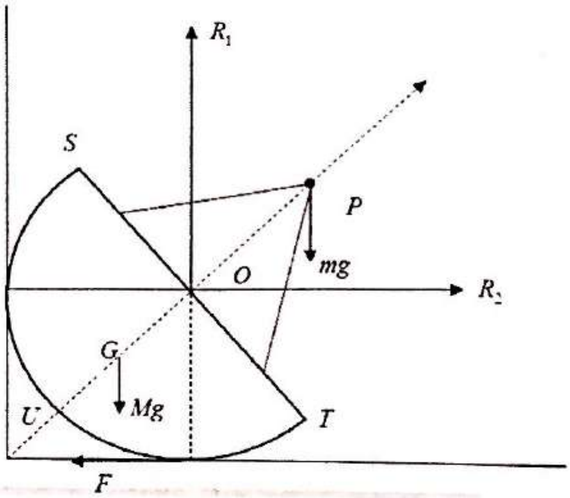
$\Leftrightarrow \quad k>\frac{1}{4}.$ (5)

25

Let $k=1$.

Then, $\overline{x_1}=\left(\frac{\pi+7}{\pi+4}\right)\frac{a}{2}$.

Let $\overline{x_2}$ be the distance to the new centre of mass from P.



Then $[(4a\rho+\pi a\ \rho)+m]\overline{x_2}=(4a\rho+\pi a\ \rho)\overline{x_1}$. (5)

$\Leftrightarrow\ [(4a\rho+\pi a\ \rho)+m]\overline{x_2}=(4a\rho+\pi a\ \rho)\left(\frac{\pi+7}{\pi+4}\right)\frac{a}{2}$

$\Leftrightarrow\ [(4a\rho+\pi a\ \rho)+m]\overline{x_2}=a\rho(\pi+7)\frac{a}{2}$

$\Leftrightarrow\ \overline{x_2}=\frac{a\rho(\pi+7)}{[(4a\rho+\pi a\ \rho)+m]}\frac{a}{2}$ (5)

To maintain equilibrium in the above position, we must have $\overline{x_2}>\frac{a}{2}$. (5)

i.e. $\frac{a\rho(\pi+7)}{[(4a\rho+\pi a\ \rho)+m]}\frac{a}{2}>\frac{a}{2}$

$\Leftrightarrow\ a\rho(\pi+7)>4a\rho+\pi a\ \rho+m$

$\Leftrightarrow\ m<3a\rho.$ (5)

20

17.(*a*) Each of the bags A, B and C contains only white balls and black balls which are identical in all respects, except for colour. The bag A contains 4 white balls and 2 black balls, the bag B contains 2 white balls and 4 black balls, and the bag C contains m white balls and $(m+1)$ black balls. A bag is chosen at random and two balls are drawn from that bag at random, one after the other, without replacement. The probability that the first ball drawn is white and the second ball drawn is black, is $\frac{5}{18}$. Find the value of m.

Also, find the probability that the bag C was chosen, given that the first ball drawn is white and the second ball drawn is black.

(*b*) The following table gives the distribution of marks obtained by a group of 100 students for their answers to a Statistics question:

Marks range	Number of students
0–2	15
2–4	25
4–6	40
6–8	15
8–10	5

Estimate the mean μ and the standard deviation σ of this distribution.

Also, estimate the coefficient of skewness κ defined by $\kappa = \frac{3(\mu - M)}{\sigma}$, where M is the median of the distribution.

(*a*) Let X = First ball drawn is white and the second ball drawn is black.

By the Law of Total Probability, (5)

$$P(X) = P(X \mid A)\,P(A) + P(X \mid B)\,P(B) + P(X \mid C)\,P(C). \quad \text{-------------(1)}$$

$$P(X \mid A) = \frac{4}{6} \times \frac{2}{5} = \frac{4}{15} \quad (10)$$

$$P(X \mid B) = \frac{2}{6} \times \frac{4}{5} = \frac{4}{15} \quad (10)$$

$$P(X \mid C) = \frac{m}{(2m+1)} \cdot \frac{m+1}{2m} = \frac{(m+1)}{2(2m+1)} \quad (10)$$

Also, $P(A) = P(B) = P(C) = \frac{1}{3}$. (5)

Since $P(X) = \frac{5}{18}$,

$$(1) \Rightarrow \quad \frac{5}{18} = \frac{4}{15} \times \frac{1}{3} + \frac{4}{15} \times \frac{1}{3} + \frac{(m+1)}{2(2m+1)} \times \frac{1}{3} \quad (10)$$

$$\Rightarrow \frac{5}{6} - \frac{8}{15} = \frac{(m+1)}{2(2m+1)} \quad (5)$$

$\Rightarrow \quad 3(2m+1) = 5(m+1)$

$\Rightarrow \quad m = 2$ (5)

60

$m = 2 \Rightarrow P(X \mid C) = \frac{3}{10}.$ (5)

By Baye's Theorem,

$P(C \mid X) = \frac{P(X \mid C)\, P(C)}{P(X)}$ (5)

$= \frac{\frac{3}{10} \times \frac{1}{3}}{\frac{5}{18}}$ (5)

$= \frac{9}{25}.$ (5)

20

(b)

Marks range	f	Mid point x	x^2 (5)	$f\,x$ (5)	$f\,x^2$ (5)
0-2	15	1	1	15	15
2-4	25	3	9	75	225
4-6	40	5	25	200	1000
6-8	15	7	49	105	735
8-10	5	9	81	45	405
	$\sum f = 100$			$\sum f\,x = 440$ (5)	$\sum f\,x^2 = 2380$ (5)

$\mu = \frac{\sum f\,x}{\sum f} = \frac{440}{100} = 4.4$ (5)

$\sigma = \sqrt{\frac{\sum f\,x^2}{\sum f} - \mu^2}$ (5)

$= \sqrt{\frac{2380}{100} - \left(\frac{44}{10}\right)^2}$ (5)

$= \sqrt{23.8 - 19.36}$ (5)

$= \sqrt{4.44}$

$\approx 2.11.$ (5)

50

$M = 4 + \frac{10}{40} \times 2$ (5)

$= 4.5.$ (5)

$\kappa = \frac{3(4.4 - 4.5)}{2.11}$ (5)

$= -\frac{0.3}{2.11}$

$\approx - \ 0.14.$ (5)

20
